only read "✓" → important slides

Pattern Recognition Techniques Applied to Biomedical Signal Processing

PhD Course- Prague, October 2003

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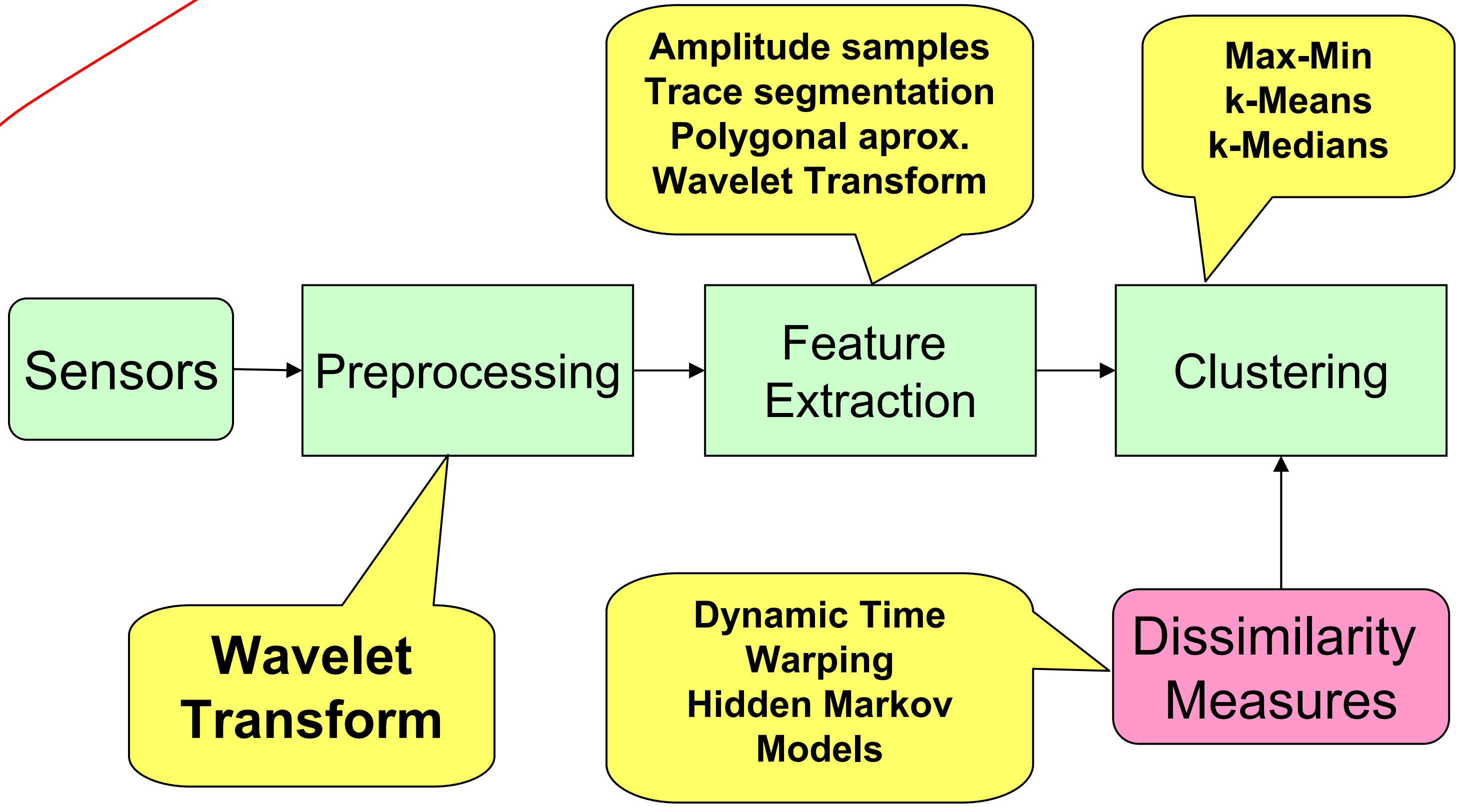
Course Syllabus

- Pattern Recognition Systems.



Schedule

06/10/2003	07/10/2003	08/10/2003	09/10/2003	10/10/2003
	14.30 •Wavelet transform	**10.30** •Feature extraction •Clustering algorithms **14.30** •Dynamic Time Warping		**15:30** •Hidden Markov Models
13/10/2003	14/10/2003	15/10/2003	16/10/2003	17/10/2003

Course Structure

- Lectures.
- Laboratory exercises.
- Assignments.

- Total Credits:1-2

Signal Processing

Introduction to Wavelets

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1.- Introduction.

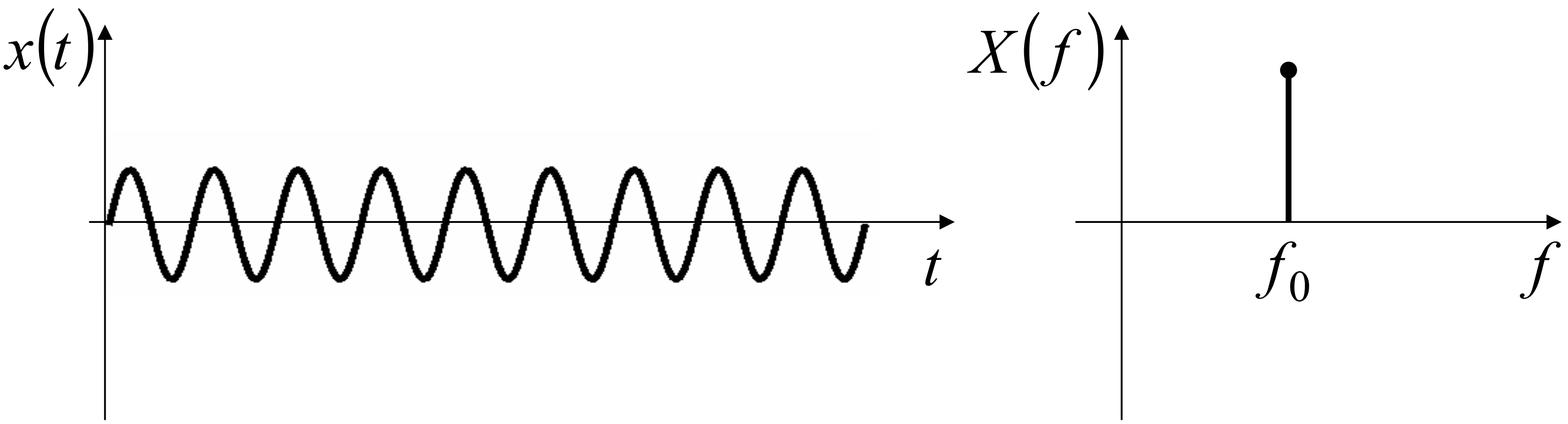
- Signal Processing.
  - Extract information from a signal.
  - Time domain is not always the best choice.
  - Frequency domain: Fourier Transform:

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

  - Easily implemented in computers: FFT.

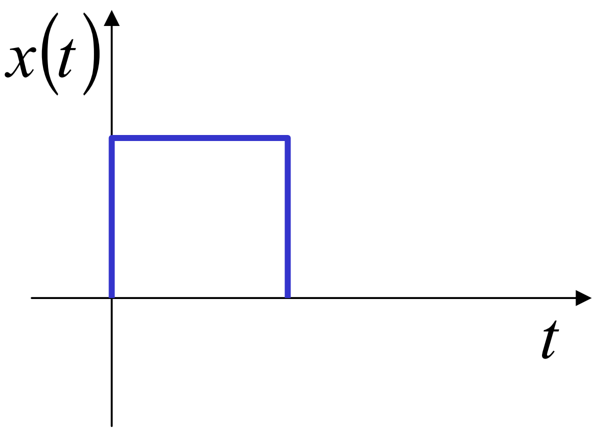
1.- Introduction.

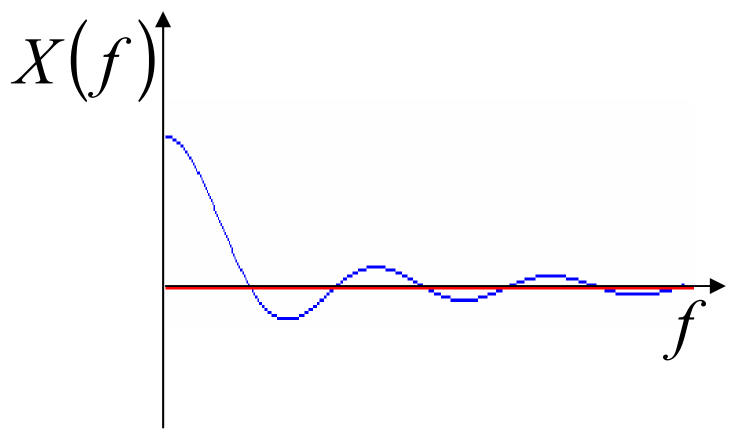
- Fourier Analysis.
  - Can not provide simultaneously time and frequency information:Time information is lost.
  - Small changes in frequency domain cause changes everywhere in the time domain.
  - For stationary signals: the frequency content does not change in time, i.e.:



1.- Introduction.

- Fourier Analysis.
  - Basis Functions are set: sinusoids.
  - Many coefficients needed to describe an “edge” or discontinuity.
  - If the signal is a sinusoid, it is better localized in the frequency domain.
  - If the signal is a square pulse, it is better localized in the time domain.

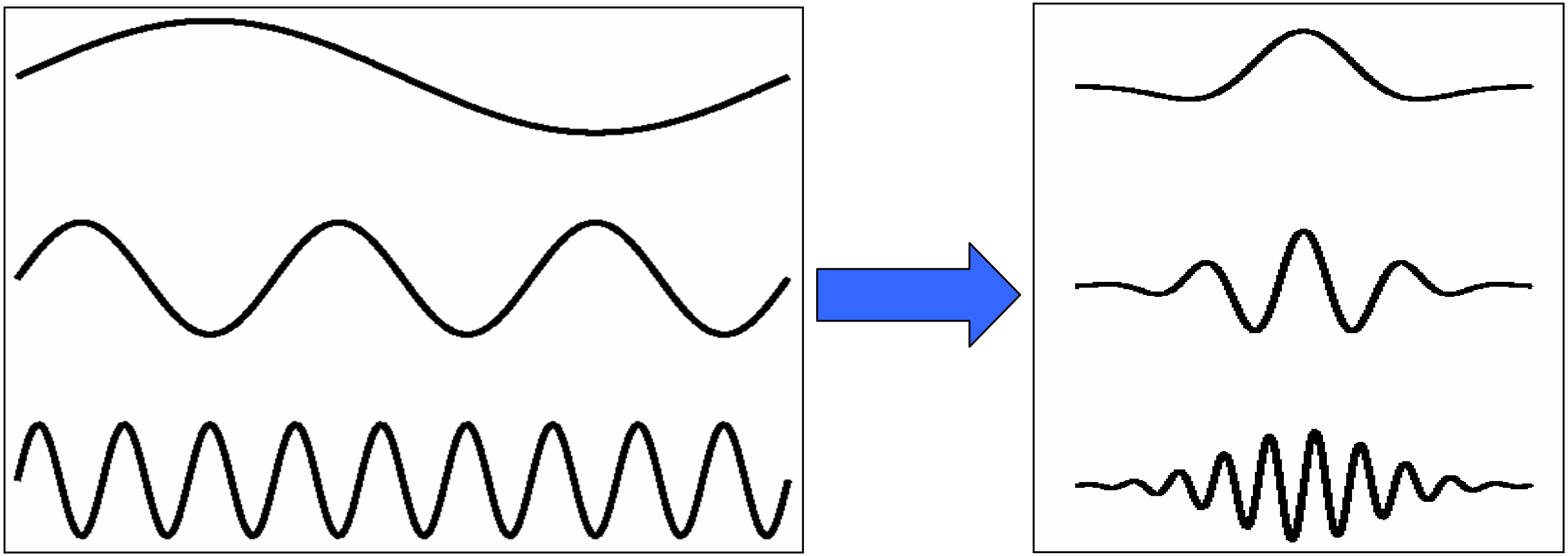




Short Time Fourier Transform.

1.- Introduction.

- Short Time Fourier Analysis.
  - Time-Frequency Analysis.
  - Windowed Fourier Transform (STFT).



FT Basis Functions

STFT Basis Functions

1.- Introduction.

- Short Time Fourier Analysis.

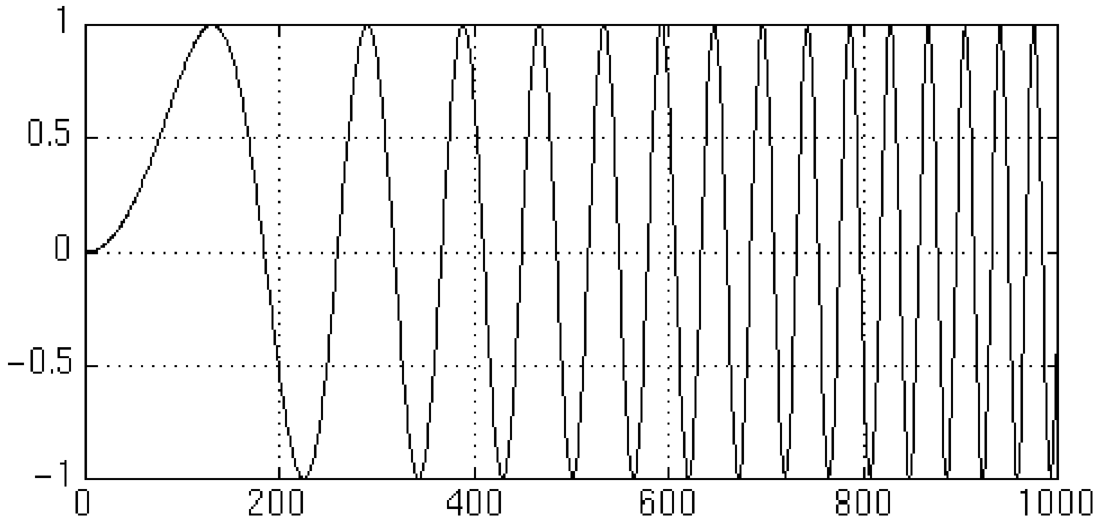
$$STFT(t_0, \omega_0) = \int f(t) g(t - t_0) \sin(\omega_0 t) dt$$

  - Discrete version very difficult to find.
  - No fast transform.
  - Fixed window size (fixed resolution):
    - Large windows for low frequencies.
    - Small windows for high frequencies.

  - Wavelet Transform.

1.- Introduction.

- Wavelet Transform.
  - Small wave.
  - Energy concentrated in time: analysis of non-stationary, transient or time-varying phenomena.
  - Allows simultaneous time and frequency analysis.



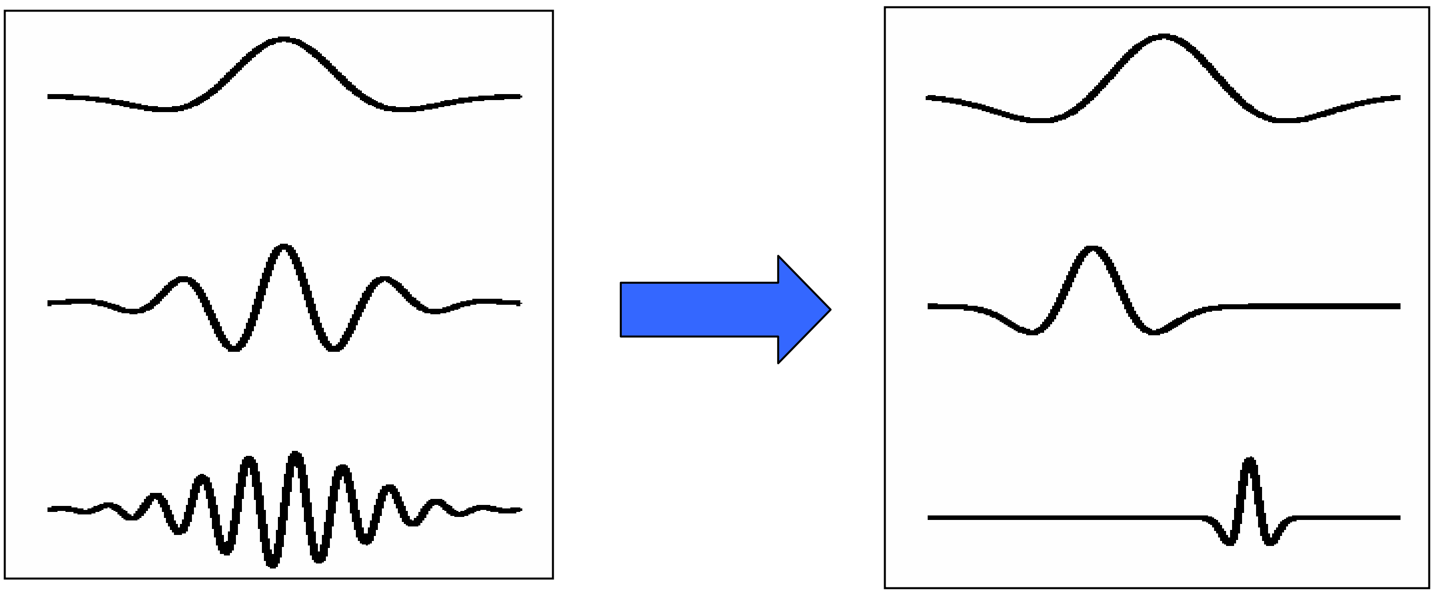
1.- Introduction.

- Wavelet Transform.
  - Fourier: localized in frequency but not in time.
  - Wavelet:
    - local in both frequency/scale (via dilations) and in time (via translations).
    - Many functions are represented in a more compact way. For example: functions with discontinuities and/or sharp spikes.
  - Fourier transform: $O(n\log_2(n))$.
  - Wavelet transform: $O(n)$.

1.- Introduction.

- Wavelet Transform.

$$WT(a,b) = \int f(t)\,\psi\left(\frac{t-b}{a}\right)dt$$



STFT Basis Functions

WT Basis Functions

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2.- Definitions.

- Mathematical Background.
  - Topic of pure mathematics, but great applicability in many fields.
  - Linear decomposition of a function:

$$f(t) = \sum_i c_i \psi_i(t)$$

$i \equiv$ Integer index for the finite or infinite sum

$c_i \equiv$ Expansion Coefficients

$\psi_i(t) \equiv$ Expansion Set

  - Decomposition Unique:

$$\{\psi_i(t)\} \equiv \text{Basis for } f(t)$$

2.- Definitions.

- Mathematical Background.
  - Basis orthogonal:

$$\langle \psi_k(t), \psi_l(t) \rangle = \int \psi_k(t)\psi_l(t)dt = 0, k \neq l$$

$$\int \psi_k(t)\psi_l(t)dt = 1, k = l$$

  - Then the coefficients can be calculated by :

$$c_k = \langle f(t), \psi_k(t) \rangle = \int f(t)\psi_k(t)dt$$

  - Since:

$$\int f(t)\psi_k(t)dt = \int \left(\sum_i c_i \psi_i(t) \right)_{i=k} \psi_k(t)dt = c_k$$

2.- Definitions.

- Mathematical Background.
  - Linear decomposition of a vector:

$$\vec{a} = (3,-2,1)$$

$$\vec{a} = 3(1,0,0) - 2(0,1,0) + 1(0,0,1)$$

$$\{(1,0,0),(0,1,0),(0,0,1)\} \qquad (3,-2,1)$$

Generator system in R^3

Independent Orthogonal

Normal

→ Basis in R^3

$(3,-2,1)$ → Coordinates in R^3

2.- Definitions.

- Mathematical Background.

If $\langle \psi_m(t), \psi_n(t) \rangle = \int \psi_m(t) \psi_n(t) dt = 0$, where $m \neq n$

$$\{\psi_i(t)\} \equiv Orthogonal\ Basis\ for\ f(t)$$

$$c_i = \langle f(t), \psi_i(t) \rangle = \int f(t) \psi_i(t)$$

Otherwise :

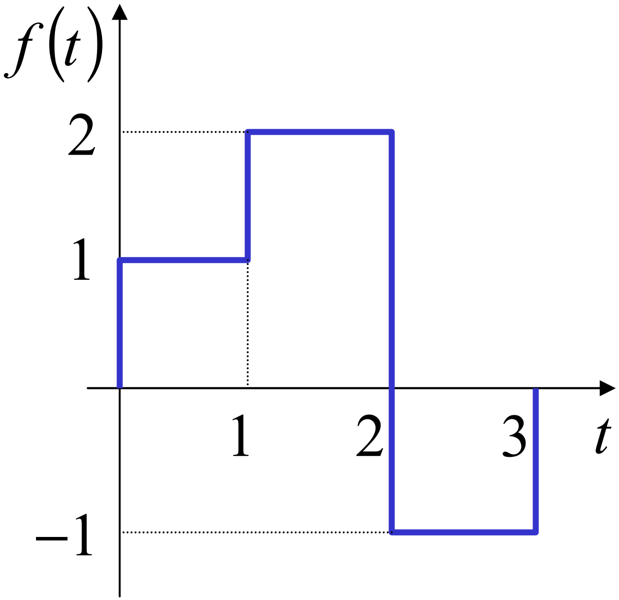
$$\{\psi_i(t), \tilde{\psi}_i(t)\} \equiv Biorthogonal\ Basis\ for\ f(t)$$

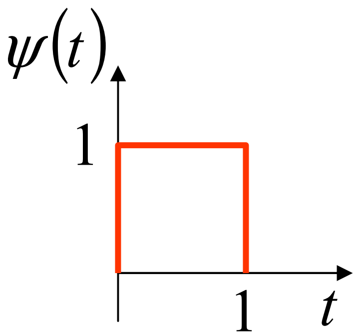
$$c_i = \langle f(t), \tilde{\psi}_i(t) \rangle$$

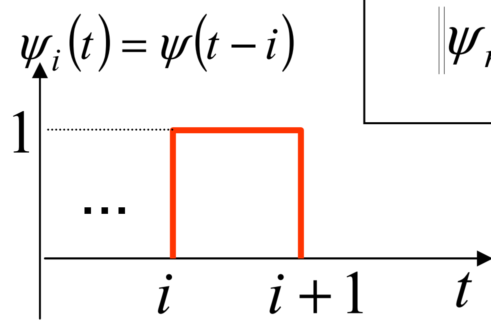
$$f(t) = \sum_i c_i \psi_i(t)$$

2.- Definitions.

- Mathematical Background.
  - Example of linear decomposition of a function:





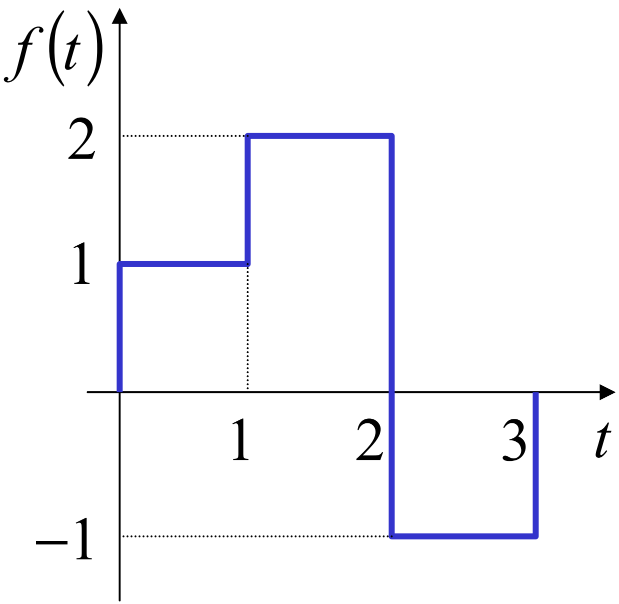


$$\int \psi_m(t)\psi_n(t)dt = \delta_{mn}$$

$$\left\|\psi_m(t)\right\| = \sqrt{\int \psi_m(t)\psi_m^*(t)dt} = 1$$

2.- Definitions.

- Mathematical Background.
  - Example of linear decomposition of a function:



$$f(t) = \psi(t) + 2\psi(t-1) - \psi(t-2)$$

$$c_0 = 1$$

$$c_1 = 2$$

$$c_2 = -1$$

$c_i = 0$, otherwise

2.- Definitions.

- Example:
  - Four dimensional space (four non-null coefficients). Synthesis Formula:

$$f(t) = \sum_i c_i \psi_i(t)$$

$$\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix} = c_0 \begin{bmatrix} \psi_0(0) \\ \psi_0(1) \\ \psi_0(2) \\ \psi_0(3) \end{bmatrix} + c_1 \begin{bmatrix} \psi_1(0) \\ \psi_1(1) \\ \psi_1(2) \\ \psi_1(3) \end{bmatrix} + c_2 \begin{bmatrix} \psi_2(0) \\ \psi_2(1) \\ \psi_2(2) \\ \psi_2(3) \end{bmatrix} + c_3 \begin{bmatrix} \psi_3(0) \\ \psi_3(1) \\ \psi_3(2) \\ \psi_3(3) \end{bmatrix}$$

2.- Definitions.

- Example:

$$\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix} = \begin{bmatrix} \psi_0(0) & \psi_1(0) & \psi_2(0) & \psi_3(0) \\ \psi_0(1) & \psi_1(1) & \psi_2(1) & \psi_3(1) \\ \psi_0(2) & \psi_1(2) & \psi_2(2) & \psi_3(2) \\ \psi_0(3) & \psi_1(3) & \psi_2(3) & \psi_3(3) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\mathbf{f} = \boldsymbol{\psi}\mathbf{c}$$

2.- Definitions.

- Example:

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} \tilde{\psi}_0(0) & \tilde{\psi}_0(1) & \tilde{\psi}_0(2) & \tilde{\psi}_0(3) \\ \tilde{\psi}_1(0) & \tilde{\psi}_1(1) & \tilde{\psi}_1(2) & \tilde{\psi}_1(3) \\ \tilde{\psi}_2(0) & \tilde{\psi}_2(1) & \tilde{\psi}_2(2) & \tilde{\psi}_2(3) \\ \tilde{\psi}_3(0) & \tilde{\psi}_3(1) & \tilde{\psi}_3(2) & \tilde{\psi}_3(3) \end{bmatrix} \begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ f(3) \end{bmatrix}$$

$$\mathbf{c} = \tilde{\boldsymbol{\psi}}^{\mathbf{T}} \mathbf{f}$$

2.- Definitions.

- Example:

$$\mathbf{c} = \tilde{\psi}^{T}\mathbf{f}$$

$$\mathbf{f} = \psi\mathbf{c}$$

$$\rightarrow \mathbf{f} = \psi\tilde{\psi}^{T}\mathbf{f} \rightarrow \tilde{\psi}^{T} = \psi^{-1}$$

Dual Basis

If $\mathbf{f}$ columns are orthonormal:

$$\psi\psi^{T} = \mathbf{I} \quad \mathbf{f} = \psi\psi^{T}\mathbf{f} \quad \tilde{\psi}^{T} = \psi^{T}$$

Single Basis

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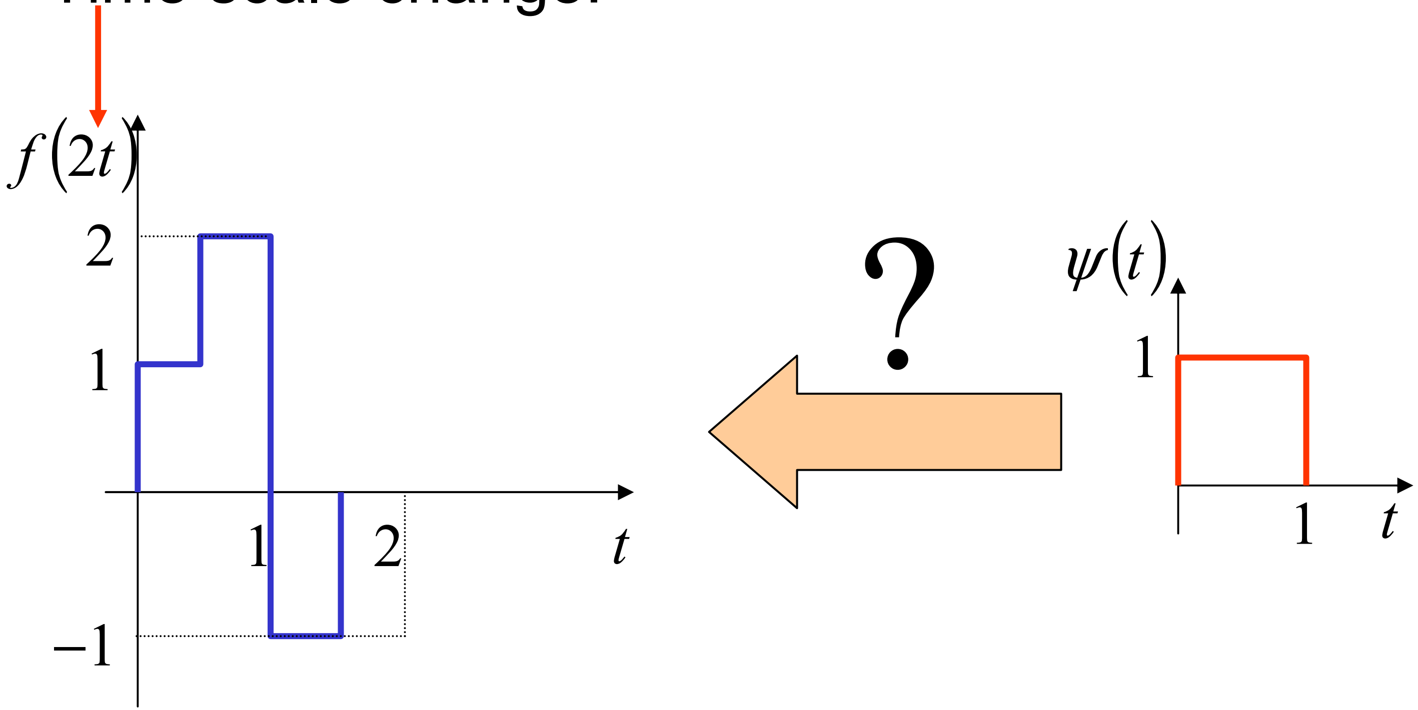
3.- Multiresolution Analysis.

- Problem definition:

Time scale change.



3.- Multiresolution Analysis.

- Multiresolution Formulation.
  - Rationale: if a set of signals can be represented by a weighted sum of $\varphi(t-k)$, a larger set (including the original), can be represented by a weighted sum of $\varphi(2t-k)$.
  - Signal analysis at different frequencies with different resolutions.
  - Good time resolution and poor frequency resolution at high frequencies.
  - Good frequency resolution and poor time resolution at low frequencies.
  - Used to define and construct orthonormal wavelet basis for L^2.
  - Two functions needed: the Wavelet and a scaling function.

3.- Multiresolution Analysis.

- Multiresolution Formulation.
  - Effect of changing the scale. Scaling function: $\varphi(t)$
  - To decompose a signal into finer and finer details.

$$f(t) \in L^2$$

Subspace : $v_0 = \underset{k}{span}\{\varphi_k(t)\} \subset L^2$

$$f(t) = \sum_k c_k \varphi_k(t) \quad \forall f(t) \in v_0$$

  - Increase the size of the subspace changing the time scale of the scaling functions:

$$\varphi_{jk}(t) = 2^{\frac{-j}{2}} \varphi\left(2^{-j} t - k\right)$$

3.- Multiresolution Analysis.

- Multiresolution Formulation.

Subspace : $v_j = span\{\varphi_k(2^j t)\} = span\{\varphi_{jk}(t)\} \subset L^2$

$$f(t) = \sum_k c_k \varphi(2^j t + k) \quad \forall f(t) \in v_j$$

- The spanned spaces are nested:

$$\ldots \subset v_{-2} \subset v_{-1} \subset v_0 \subset v_1 \subset v_2 \subset \ldots \subset L^2$$

$$f(t) \in v_j \Leftrightarrow f(2t) \in v_{j+1}$$

$$\varphi(t) = \sum_n h(n)\sqrt{2}\varphi(2t - n), n \in Z$$

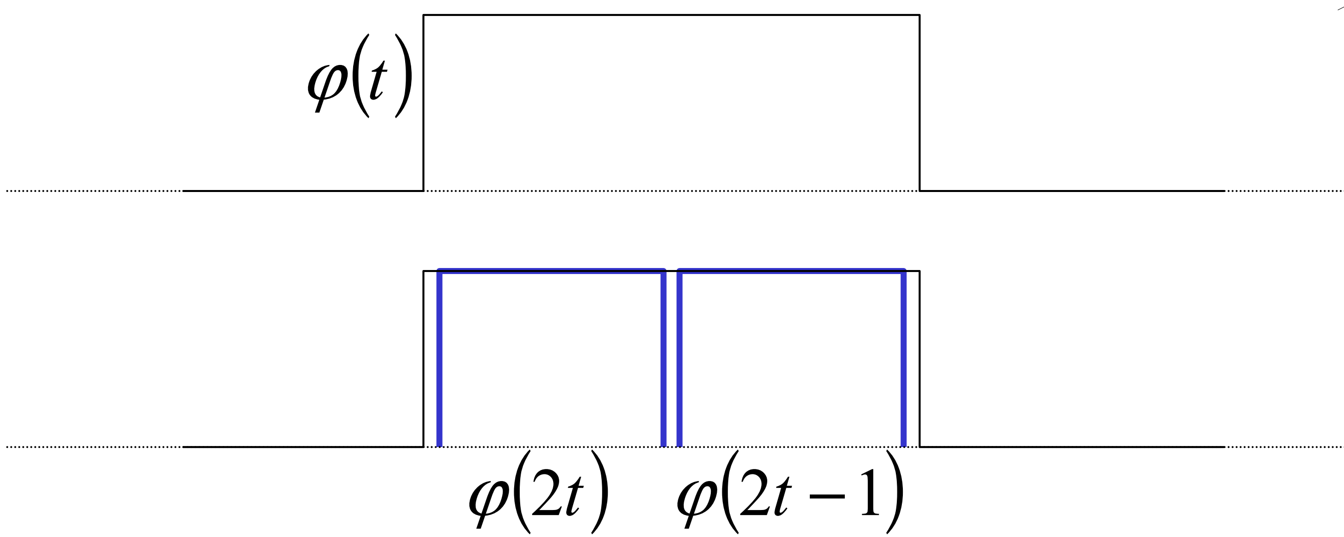
3.- Multiresolution Analysis.

- Multiresolution Formulation.
  - Example: Haar Scaling Function.

$$h(0)=\frac{1}{\sqrt{2}}$$

$$\varphi(t)=\varphi(2t)+\varphi(2t-1)$$

$$h(1)=\frac{1}{\sqrt{2}}$$



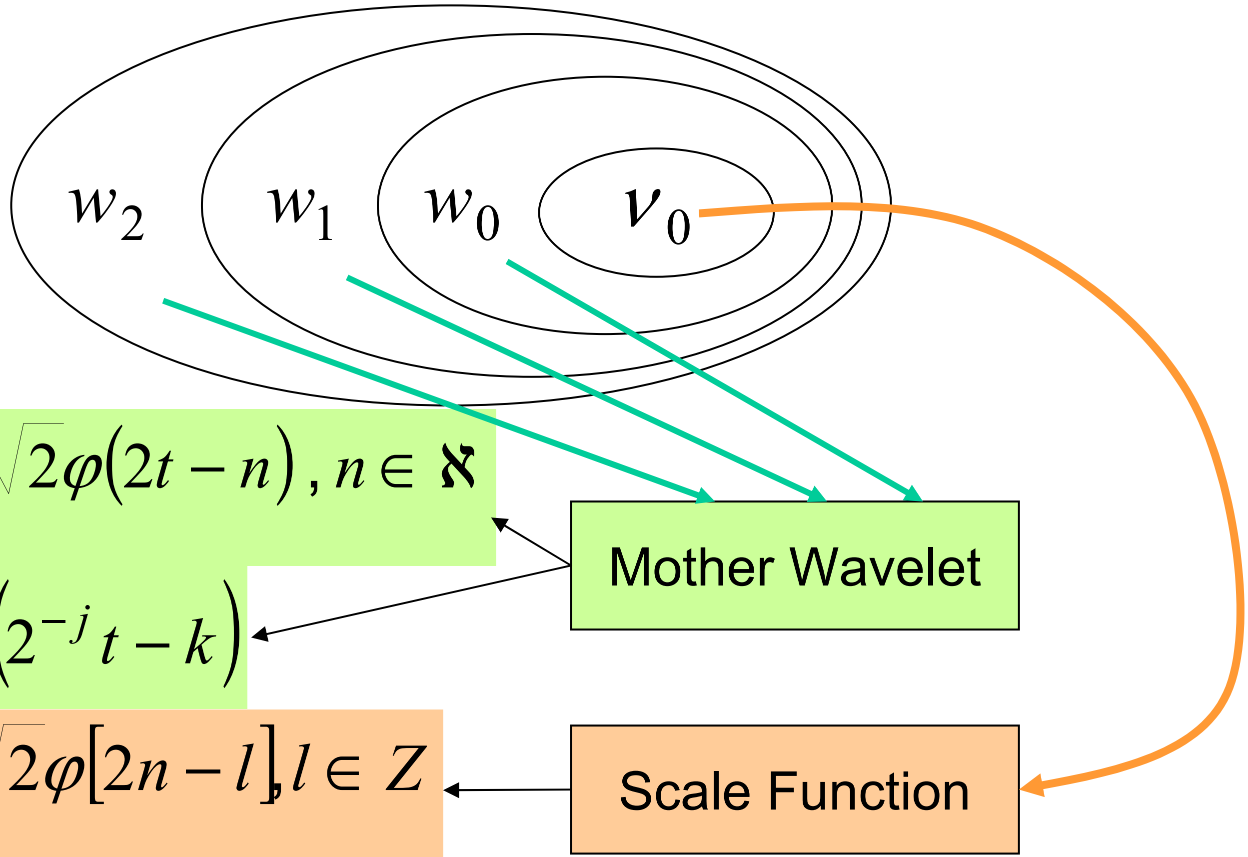
3.- Multiresolution Analysis.

- Multiresolution Formulation.
  - Increasing j, the size of the subspace spanned by the scaling function, is also increased.
  - Wavelets span the differences between spaces w_i. More suitable to describe signals.
  - Wavelets and scaling functions should be orthogonal: simple calculation of coefficients.

$$v_1 = v_0 \oplus w_0$$
$$v_2 = v_0 \oplus w_0 \oplus w_1$$
$$\ldots$$
$$L^2 = v_0 \oplus w_0 \oplus w_1 \oplus \ldots$$

3.- Multiresolution Analysis.

- Multiresolution Formulation.



$$\psi(t) = \sum_n h_1[n]\sqrt{2}\varphi(2t-n),\, n \in \aleph$$

$$\psi_{jk}(t) = 2^{\frac{-j}{2}}\psi\left(2^{-j}t-k\right)$$

$$\varphi[n] = \sum_l h[l]\sqrt{2}\varphi[2n-l], l \in Z$$

3.- Multiresolution Analysis.

- Multiresolution Formulation.

$h(n) \equiv$ scaling function coefficients

$$h_1(n) = (-1)^n h(1-n)$$

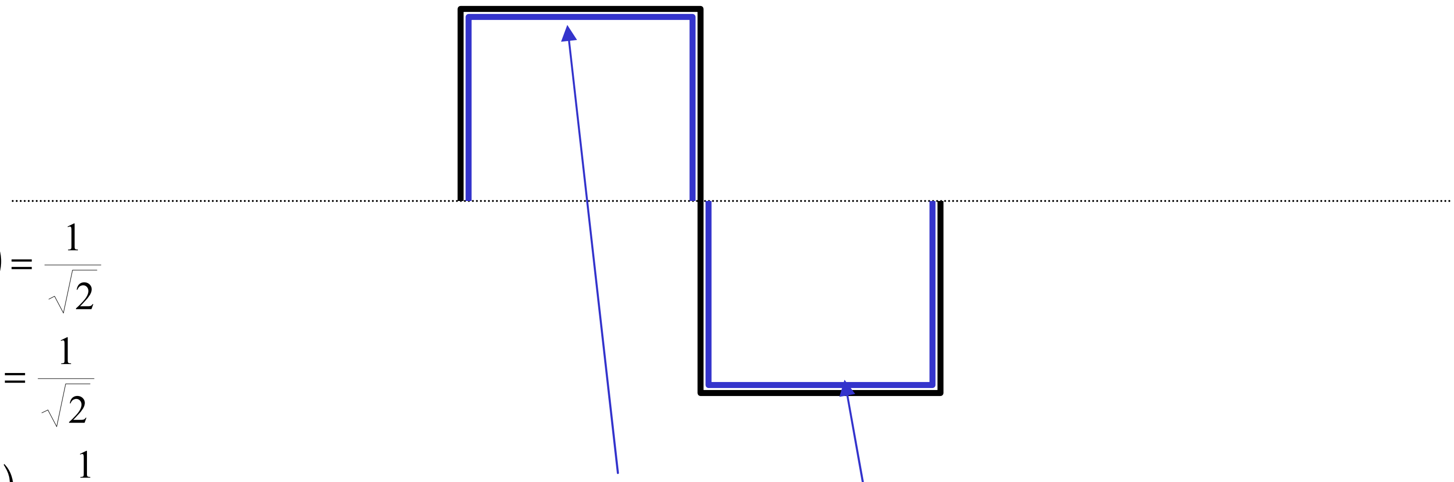
$\sqrt{2} \equiv$ accounts for norm = 1

$$\left\| g(t) \right\| = \sqrt{\int_{-\infty}^{+\infty} g(t)^2 \, dt} = \sqrt{\int_{0}^{t_1} g(t)^2 \, dt}$$

$$\left\| g(2t) \right\| = \sqrt{\int_{-\infty}^{+\infty} g(2t)^2 \, dt} = \sqrt{\int_{0}^{\frac{t_1}{2}} g(2t)^2 \, dt} \underset{u=2t}{=} \sqrt{\int_{0}^{t_1} g(u)^2 \, \frac{du}{2}} = \frac{1}{\sqrt{2}} \left\| g(t) \right\|$$

3.- Multiresolution Analysis.

- Multiresolution Formulation.
  - Example: Haar Wavelet (the oldest and simplest!).



$$h(0) = \frac{1}{\sqrt{2}}$$

$$h(1) = \frac{1}{\sqrt{2}}$$

$$h_1(0) = \frac{1}{\sqrt{2}}$$

$$h_1(1) = \frac{-1}{\sqrt{2}}$$

$$\psi(t) = \phi(2t) - \phi(2t-1)$$

3.- Multiresolution Analysis.

- Multiresolution Formulation.
  - Haar Wavelet and Scaling Function:

$$\psi(t)=\begin{cases}1, t\in\left[0,\tfrac{1}{2}\right[\\ -1, t\in\left[\tfrac{1}{2},1\right[\end{cases}$$

$$\varphi(t)=1, t\in[0,1[$$

$$\psi_{jk}(t)=2^{\frac{-j}{2}}\psi\left(2^{-j}t-k\right)$$

$$\{\psi_{jk}, j\in Z, k\in Z\}$$

$$\{\varphi_{j_0 k},\psi_{jk}, j\geq j_0, k\in Z\}$$

→ Orthonormal basis of L^2

3.- Multiresolution Analysis.

- Multiresolution Formulation.

$$L^2 = v_0 \oplus w_0 \oplus w_1 \oplus \ldots$$

$$f(t) = \sum_{k=-\infty}^{\infty} c_k \varphi_k(t) + \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} d_{jk} \psi_{jk}(t)$$

Scale Function

Wavelet

$$c_k = \langle f(t), \varphi_k(t) \rangle$$

Approximation

$$d_{jk} = \langle f(t), \psi_{jk}(t) \rangle$$

Details

Index

4.- Discrete Wavelet Transform.

- Wavelet Transform:
  - Building blocks to represent a signal.
  - Two dimensional expansion. Wavelet set:

$$\psi_{jk}(t)$$

$$f(t) = \sum_k \sum_j c_{jk} \psi_{jk}(t)$$

$c_{jk} \equiv$ Discrete Wavelet Transform of $f(t)$

  - Time-Frequency localization of the signal.
  - The calculation of the coefficients can be done efficiently.

4.- Discrete Wavelet Transform.

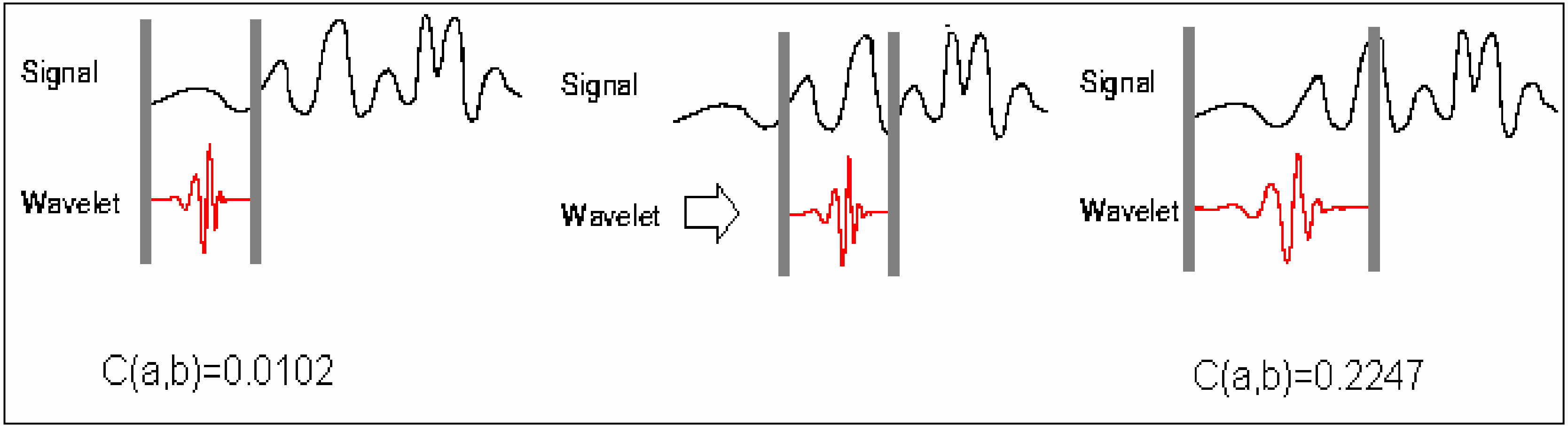
- Wavelet Transform:
  - Wavelet Systems are generated from a mother Wavelet by scaling and translation.

$$\psi_{jk}(t) = 2^{\frac{-j}{2}} \psi\left(2^{-j} t - k\right), \quad j,k \in Z$$

  - Multiresolution conditions satisfied.
  - Filter bank: the lower resolution coefficients can be calculated from the higher resolution coefficients by a tree-structured algorithm.

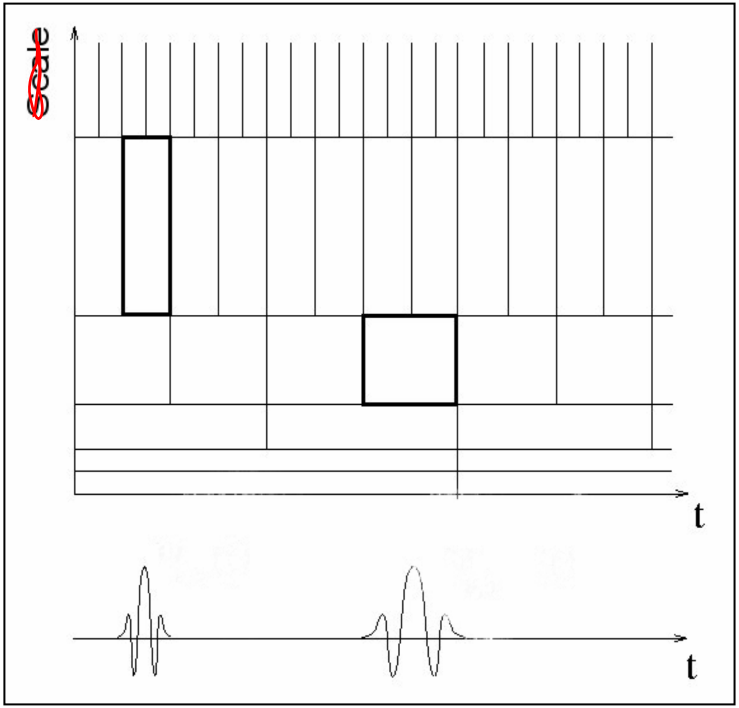
4.- Discrete Wavelet Transform.

- Wavelet Transform:
  - The effect of the Wavelet Transform is a convolution to measure the similarity between a translated and scaled version of the Wavelet and the signal under analysis.



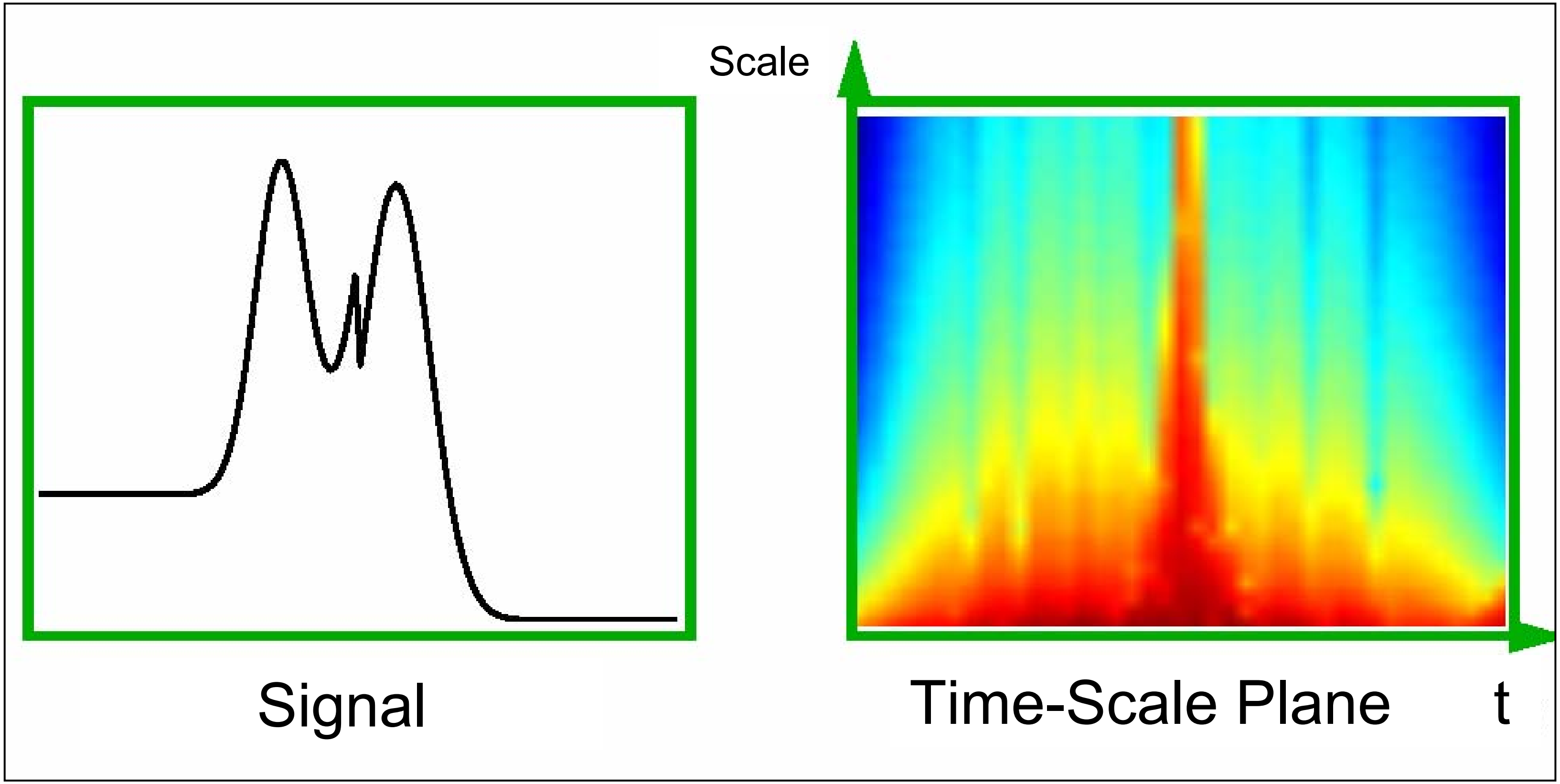
4.- Discrete Wavelet Transform.

- Wavelet Transform:
  - Wavelet Transform representation: Time-Scale Plane:



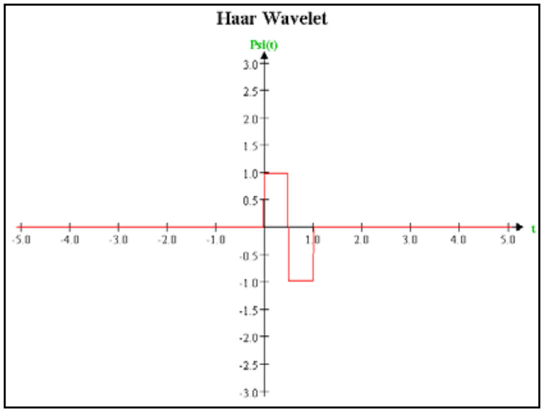
4.- Discrete Wavelet Transform.

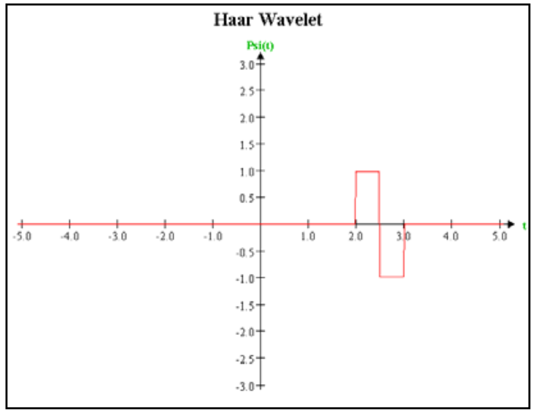
- Wavelet Transform:

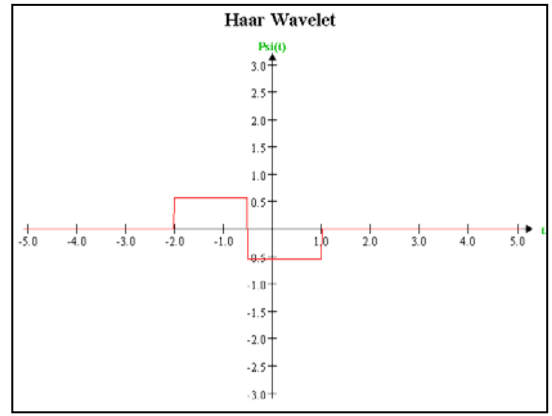


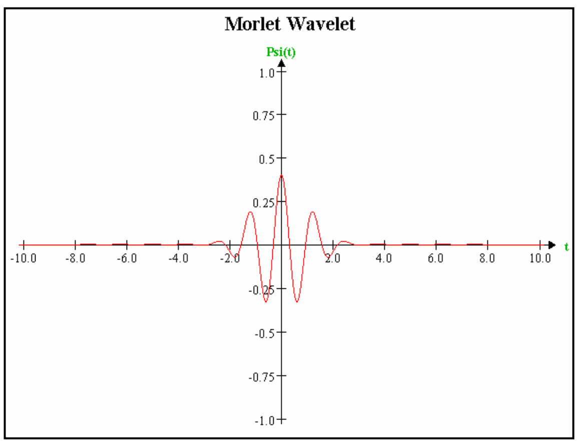
4.- Discrete Wavelet Transform.

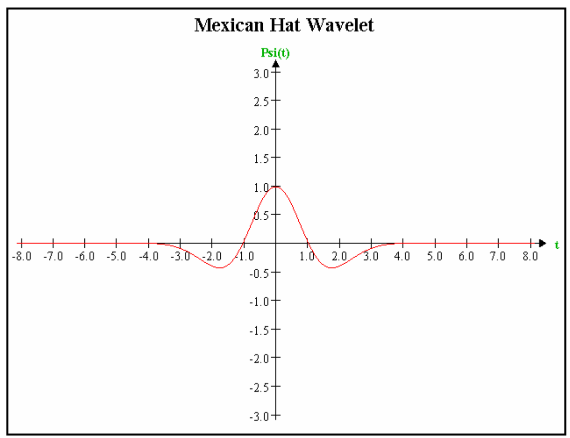
- Some Mother Wavelets:











Compact Support

4.- Discrete Wavelet Transform.

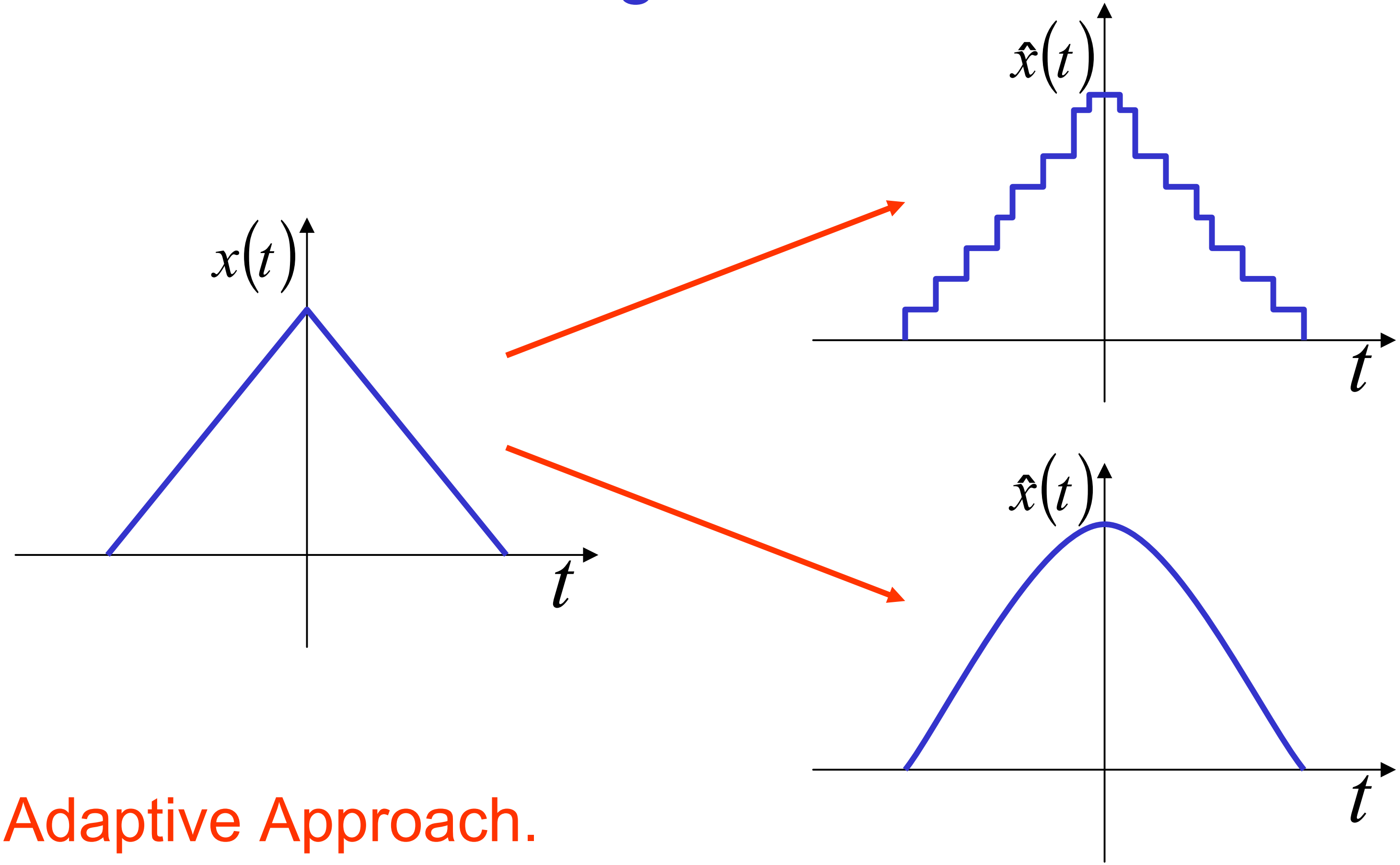
- Effect of choosing different Mother Wavelets:

The Wavelet transform in essence performs a correlation analysis, therefore the output is expected to be maximal when the input signal most resembles the mother wavelet.

Depending on the application, mother wavelet should be as similar as possible to the signal or to the signal portion under analysis. For example, the Mexican Hat Wavelet is the optimal detector to find a Gaussian.

4.- Discrete Wavelet Transform.

- Effect of choosing different Mother Wavelets:



Adaptive Approach.

4.- Discrete Wavelet Transform.

- **Wavelet Transform Calculation:**
  1. Choose a mother wavelet.
  2. Given two values of j and k, calculate the coefficient according to:

$$WT(f(t)) = c_{jk} = \int_{-\infty}^{\infty} f(t)\psi_{jk}(t)dt$$

$$\psi_{jk}(t) = \frac{1}{\sqrt{j}}\psi\left(\frac{t-k}{j}\right)$$

  3. Translate the Wavelet and repeat step 2 until the whole signal is analyzed.
  4. Scale the Wavelet and repeat steps 2 and 3.

4.- Discrete Wavelet Transform.

- Discrete Wavelet Transform Calculation:
  - Follow steps 1-4 defined previously.
  - Change the expressions:

$$DWT = c_{jk} = \sum_{n} f[n]\psi_{jk}[n]$$

$$\psi_{jk}[n] = 2^{\frac{-j}{2}}\psi\left[2^{-j}n - k\right]$$

$$DWT^{-1} = f[n] = \sum_{j\in N}\sum_{k\in N} c_{jk}\psi_{jk}[n]$$

4.- Discrete Wavelet Transform.

- Discrete Wavelet Transform Calculation:
  - Using Multiresolution Analysis:

$$DWT\left(f[n]\right) = \begin{cases} c_k = \left\langle f[n], \varphi_k[n] \right\rangle = \sum_n f[n]\varphi_k[n] \\ d_{jk} = \left\langle f[n], \psi_{jk}[n] \right\rangle = \sum_n f[n]\psi_{jk}[n] \end{cases}$$

$$\psi[n] = \sum_l h_1[l]\sqrt{2}\varphi[2n-l], l \in Z$$

$$h_1[l] = (-1)^l h[1-l]$$

$$\varphi[n] = \sum_l h[l]\sqrt{2}\varphi[2n-l], l \in Z$$

4.- Discrete Wavelet Transform.

- Discrete Wavelet Transform Calculation:
  - Using Multiresolution Analysis:

$$DWT^{-1} = IDWT = f[n] = \sum_{k=-\infty}^{\infty} c_k \varphi_k[n] + \sum_{j=0}^{\infty} \sum_{k=-\infty}^{\infty} d_{jk} \psi_{jk}[n]$$

$$DWT^{-1} = IDWT = f[n] = \sum_{k=-\infty}^{\infty} c_{j_0 k} \varphi_{j_0 k}[n] + \sum_{j=j_0}^{\infty} \sum_{k=-\infty}^{\infty} d_{jk} \psi_{jk}[n]$$

Dyadic grid:

$$j = 0 \Rightarrow k = 0,2,4,\ldots$$

$$j = 1 \Rightarrow k = 0,4,8,\ldots$$

$$\ldots$$

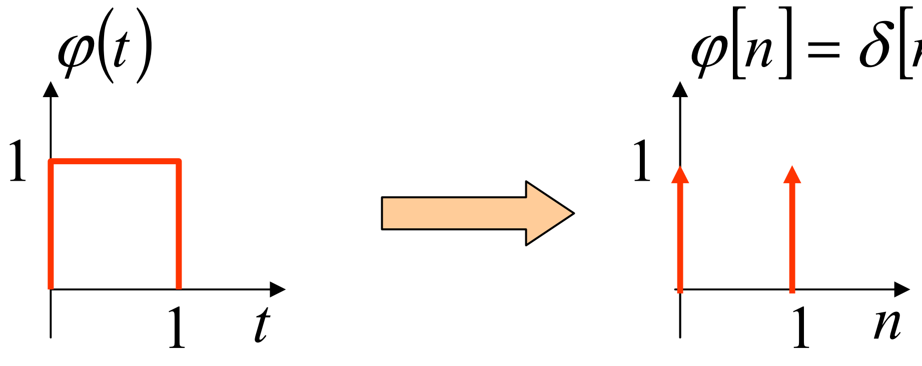
4.- Discrete Wavelet Transform.

- Discrete Wavelet Transform Calculation:
  - Example:

$$f[n] = \{1,1,1,2,3,2,1,1,1\}$$

  - Haar Scaling Function:

$$\varphi[n] = \delta[n] + \delta[n-1]$$



4.- Discrete Wavelet Transform.

- Discrete Wavelet Transform Calculation:
  - Haar Wavelet:

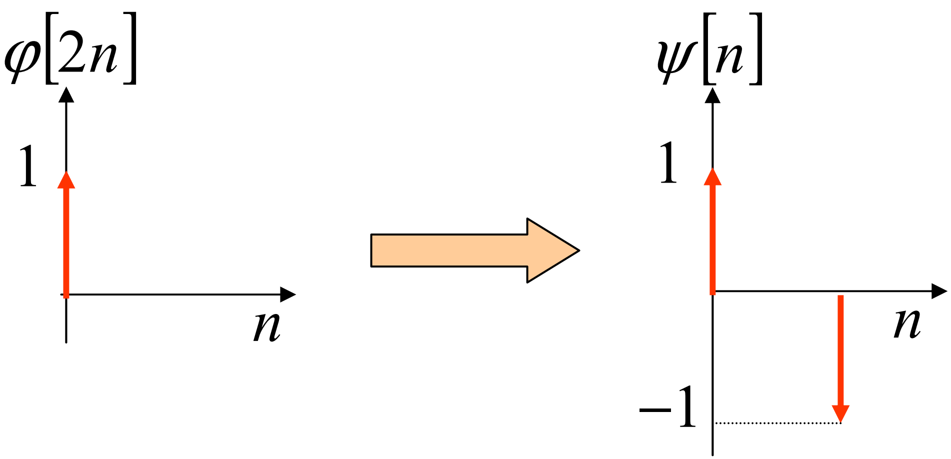
$$h(0)=\frac{1}{\sqrt{2}}$$

$$h(1)=\frac{1}{\sqrt{2}}$$

$$h_1(0)=\frac{1}{\sqrt{2}}$$

$$h_1(1)=\frac{-1}{\sqrt{2}}$$

$$\psi[n]=\varphi[2n]-\varphi[2n-1]$$



4.- Discrete Wavelet Transform.

- Discrete Wavelet Transform Calculation:

$$DWT(f[n]) = \begin{cases} c_k = \langle f[n], \varphi_k[n] \rangle = \sum_n f[n] \varphi_k[n] \\ d_{jk} = \langle f[n], \psi_{jk}[n] \rangle = \sum_n f[n] \psi_{jk}[n] \end{cases}$$

$$f[n] = \{1,1,1,2,3,2,1,1\}$$

$$\psi[n] = \delta[n] - \delta[n-1]$$

For $j = 0, k = 0,2,4,6$

$$d_{00} = f[n](\delta[n] - \delta[n-1]) = f[0] - f[1] = 0$$

$$d_{02} = f[n](\delta[n-2] - \delta[n-3]) = f[2] - f[3] = -1$$

$$d_{04} = f[4] - f[5] = 3 - 2 = 1$$

$$d_{06} = f[6] - f[7] = 1 - 1 = 0$$

4.- Discrete Wavelet Transform.

- Discrete Wavelet Transform Calculation:

$$f[n] = \{1,1,1,2,3,2,1,1\}$$

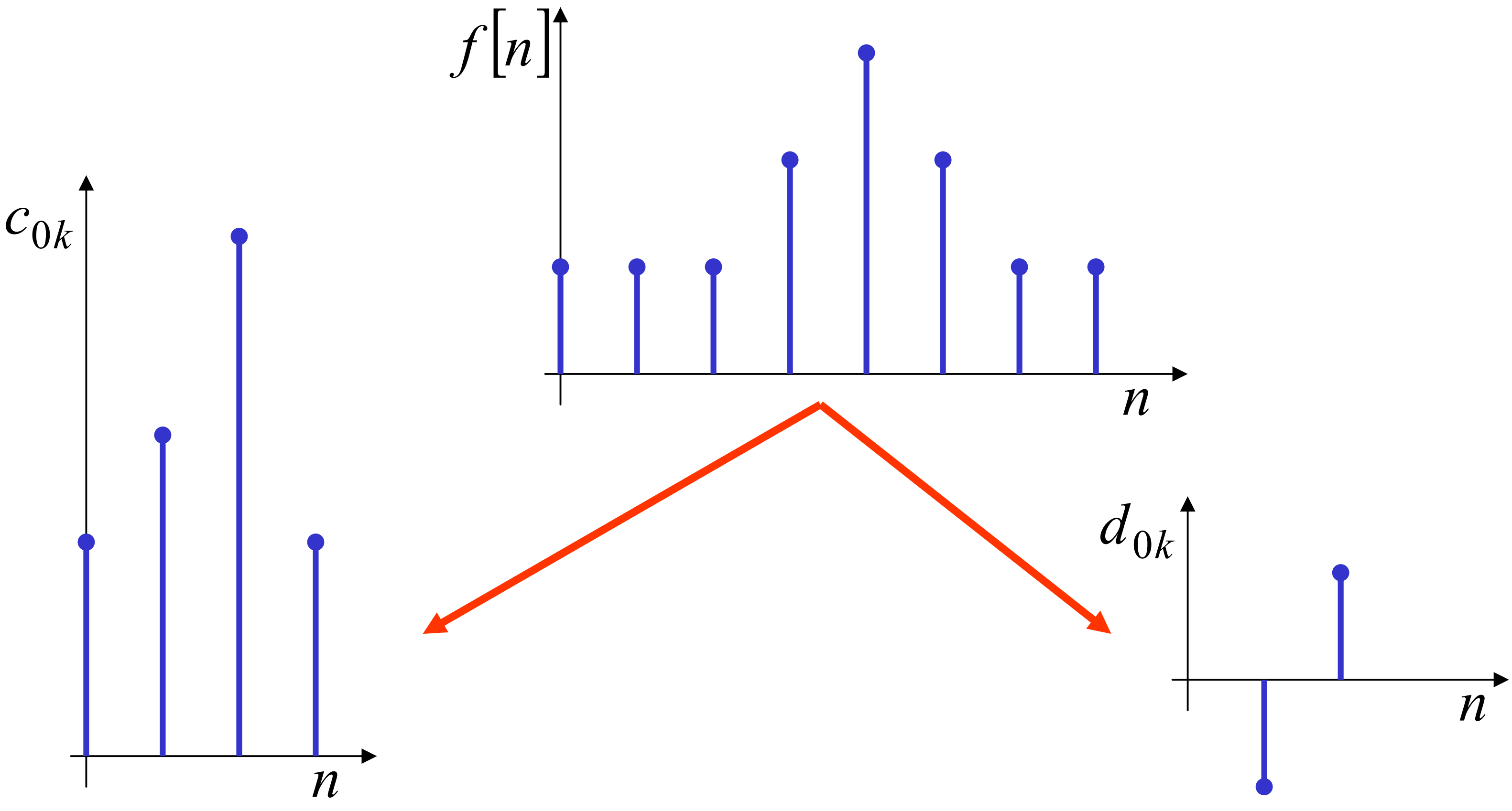
$$\varphi[n] = \delta[n] + \delta[n-1]$$

For $j = 0, k = 0,2,4,6$

$$c_0 = f[n](\delta[n] + \delta[n-1]) = f[0] + f[1] = 2$$
$$c_2 = f[n](\delta[n-2] + \delta[n-3]) = f[2] + f[3] = 3$$
$$c_4 = f[n](\delta[n-4] + \delta[n-5]) = f[4] + f[5] = 5$$
$$c_6 = f[n](\delta[n-6] + \delta[n-7]) = f[6] + f[7] = 2$$

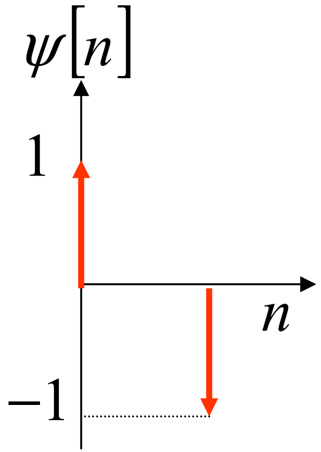
4.- Discrete Wavelet Transform.

- Discrete Wavelet Transform Calculation:

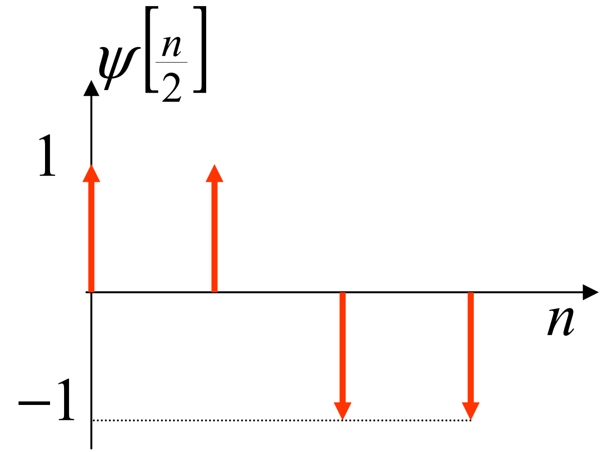


4.- Discrete Wavelet Transform.

- Discrete Wavelet Transform Calculation:



$$\psi_{jk}[n] = 2^{\frac{-j}{2}} \psi\left[2^{-j} n - k\right]$$



For $j = 1, k = 0,4$

$$d_{10} = f[n]\left(\delta[n] + \delta[n-1] - \delta[n-2] - \delta[n-3]\right) =$$

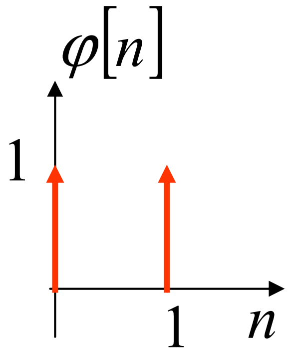
$$f[0] + f[1] - f[2] - f[3] = 1 + 1 - 1 - 2 = -1$$

$$d_{14} = f[n]\left(\delta[n-4] + \delta[n-5] - \delta[n-6] - \delta[n-7]\right) =$$

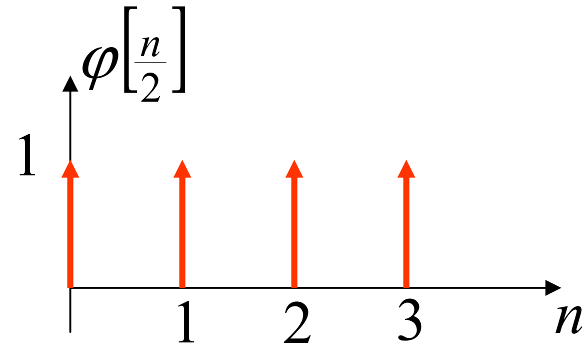
$$f[4] + f[5] - f[6] - f[7] = 3 + 2 - 1 - 1 = 3$$

4.- Discrete Wavelet Transform.

- Discrete Wavelet Transform Calculation:



$$\varphi_{jk}[n] = 2^{\frac{-j}{2}} \varphi\left[2^{-j} n - k\right]$$



For $j = 1, k = 0,4$

$$c_{10} = f[n]\left(\delta[n] + \delta[n-1] + \delta[n-2] + \delta[n-3]\right) = 1+1+1+2 = 5$$

$$c_{14} = f[n]\left(\delta[n-4] + \delta[n-5] + \delta[n-6] + \delta[n-7]\right) = 3+2+1+1 = 7$$

4.- Discrete Wavelet Transform.

- Discrete Wavelet Transform Calculation:

Complete Decomposition (without factor $2^{\frac{-j}{2}}$):

$f[n]$	1	1	1	2	3	2	1	1	
c_{0k}	2	3	5	2	0	-1	1	0	d_{0k}
c_{1k}	5	7	-1	3	0	-1	1	0	d_{1k}
c_{2k}	12	-2	-1	3	0	-1	1	0	d_{2k}

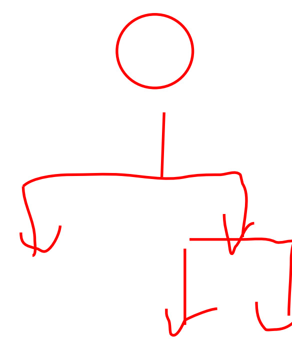
4.- Discrete Wavelet Transform.

- Discrete Wavelet Transform Calculation:

Reconstruction:

$$f[n] = \sum_{k=-\infty}^{\infty} c_{j_0 k}\varphi_{j_0 k}[n] + \sum_{j=j_0}^{\infty}\sum_{k=-\infty}^{\infty} d_{jk}\psi_{jk}[n]$$

From level $j=0$:



$$f[n] = \sum_{k=0,2,4,6} c_{0k}\varphi_{0k}[n] + \sum_{k=0,2,4,6} d_{0k}\psi_{0k}[n] =$$

$$(2,2,3,3,5,5,2,2) + (0,0,-1,1,1,-1,0,0) =$$

$$(2,2,2,4,6,4,2,2) \Rightarrow 2f[n]$$

4.- Discrete Wavelet Transform.

- Discrete Wavelet Transform Calculation:

Wavelet	0	1	2	3	4	5
Haar	1.0	1.0				
Daubechies-4	$\frac{1}{4}\left(1+\sqrt{3}\right)$	$\frac{1}{4}\left(3+\sqrt{3}\right)$	$\frac{1}{4}\left(3-\sqrt{3}\right)$	$\frac{1}{4}\left(1-\sqrt{3}\right)$		
Daubechies-6	0.332671	0.806891	0.459877	-0.135011	-0.085441	0.035226

Order of the Wavelet: number of nonzero coefficients.

Value of wavelet coefficients is determined by constraints of orthogonality and normalization.

4.- Discrete Wavelet Transform.

- Discrete Wavelet Transform Calculation:
  - Calculation to be implemented in a computer.
  - Is there any method, such as FFT, for DWT efficient calculation?

Pyramid Algorithm

  - Basic Idea: DWT (direct and inverse) can be thought of as a filtering process.

4.- Discrete Wavelet Transform.

- Pyramid Algorithm:

Filtering: $$f[n]*h[n]=\sum_{k=-\infty}^{\infty} f[k]h[n-k]$$

$$DWT(f[n])=\begin{cases} c_k=\langle f[n],\varphi_k[n]\rangle=\sum_n f[n]\varphi_k[n] \\ d_{jk}=\langle f[n],\psi_{jk}[n]\rangle=\sum_n f[n]\psi_{jk}[n] \end{cases}$$

Signal is passed through a half-band lowpass filter

Signal is passed through a half-band highpass filter

Sampling Frequency: 2π

Bandwidth: π

4.- Discrete Wavelet Transform.

- Pyramid Algorithm:
  - After filtering, half of the samples can be eliminated: subsample the signal by two.
    - Subsampling: Scale is doubled.
    - Filtering: Resolution is halved.
  - Decomposition is obtained by successive highpass and lowpass filtering.

$$y_{high}[n] = \sum_{k=-\infty}^{\infty} f[k] g[2n-k]$$

$$y_{low}[n] = \sum_{k=-\infty}^{\infty} f[k] h[2n-k]$$

4.- Discrete Wavelet Transform.

- Pyramid Algorithm:
  - Filters h and g are not independent.
  - Relation between filters:

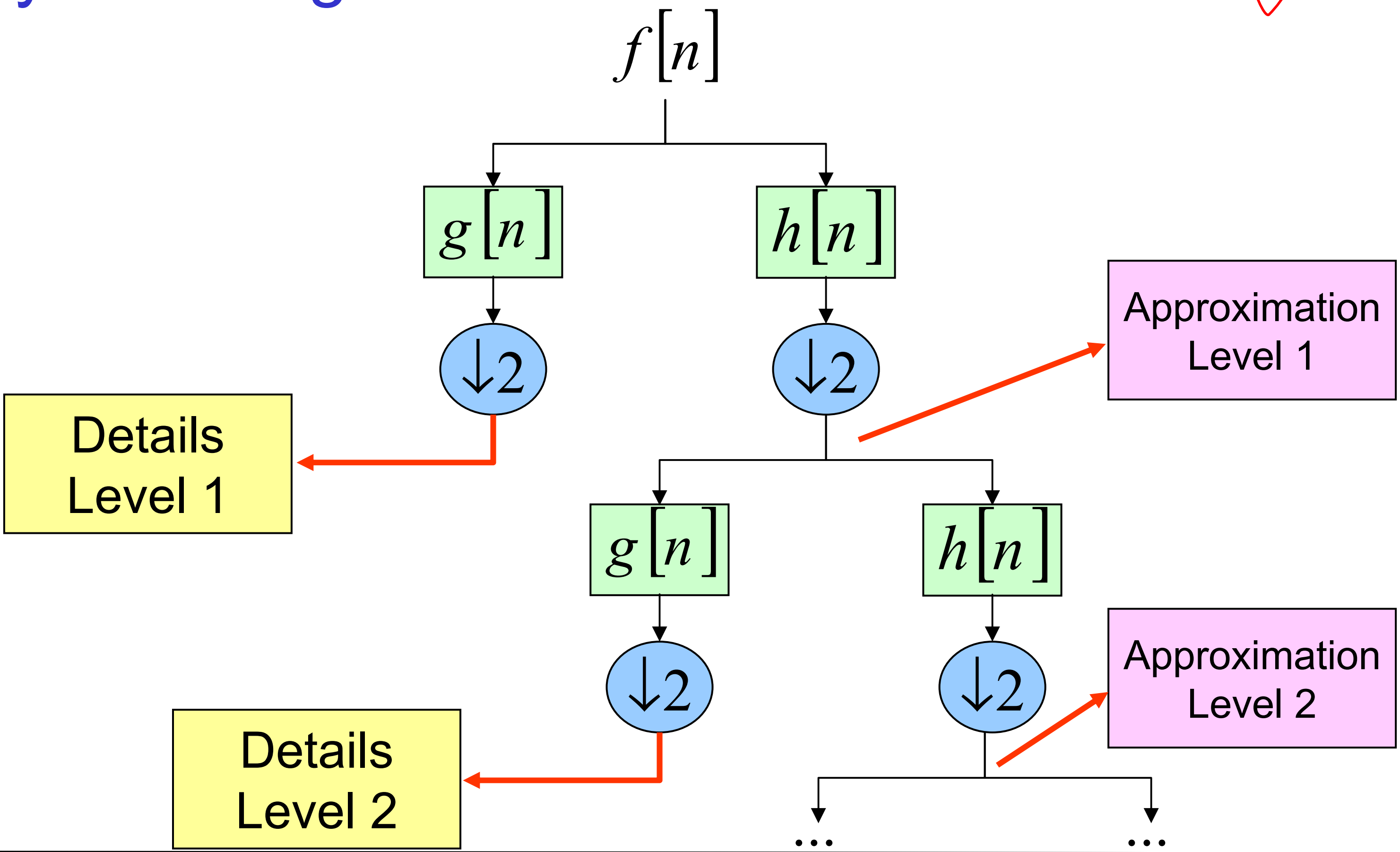
$$g[L-1-n] = (-1)^n h[n] \quad (L \equiv \text{filter length})$$

Quadrature Mirror Filters.

  - Signal length must be a power of 2 (due to downsampling by 2).
  - Maximum decomposition level depends on signal length.

4.- Discrete Wavelet Transform.

- Pyramid Algorithm:



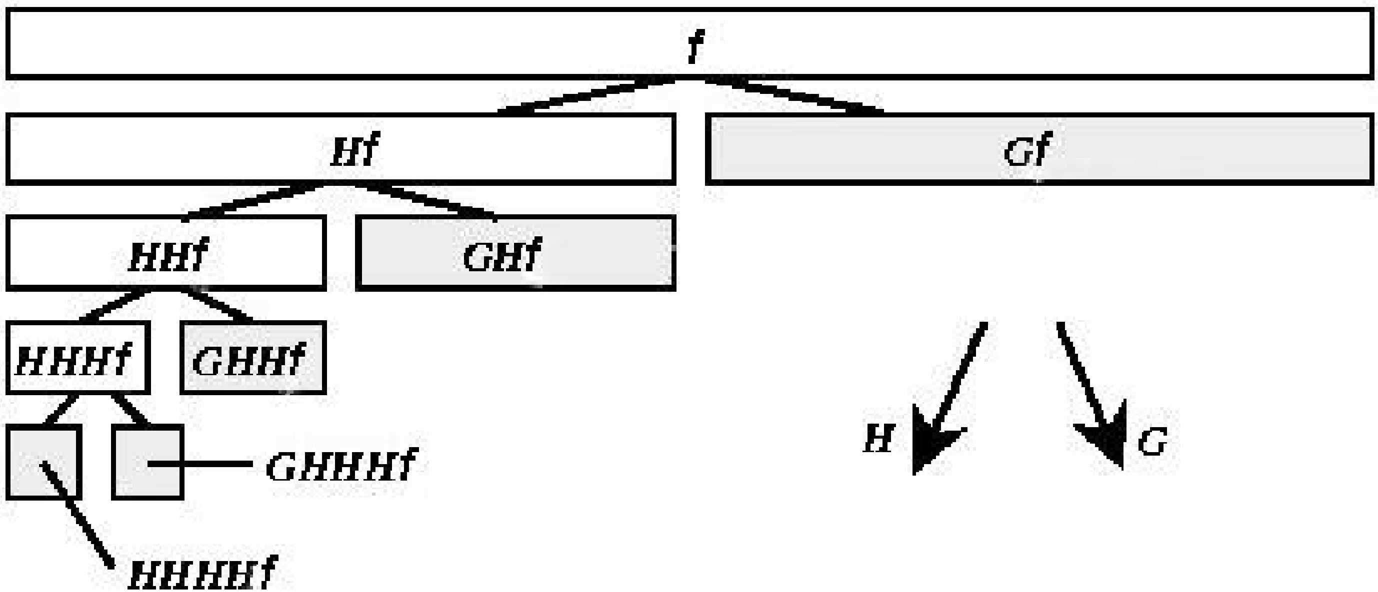
4.- Discrete Wavelet Transform.

- Pyramid Algorithm:

  Operators notation.

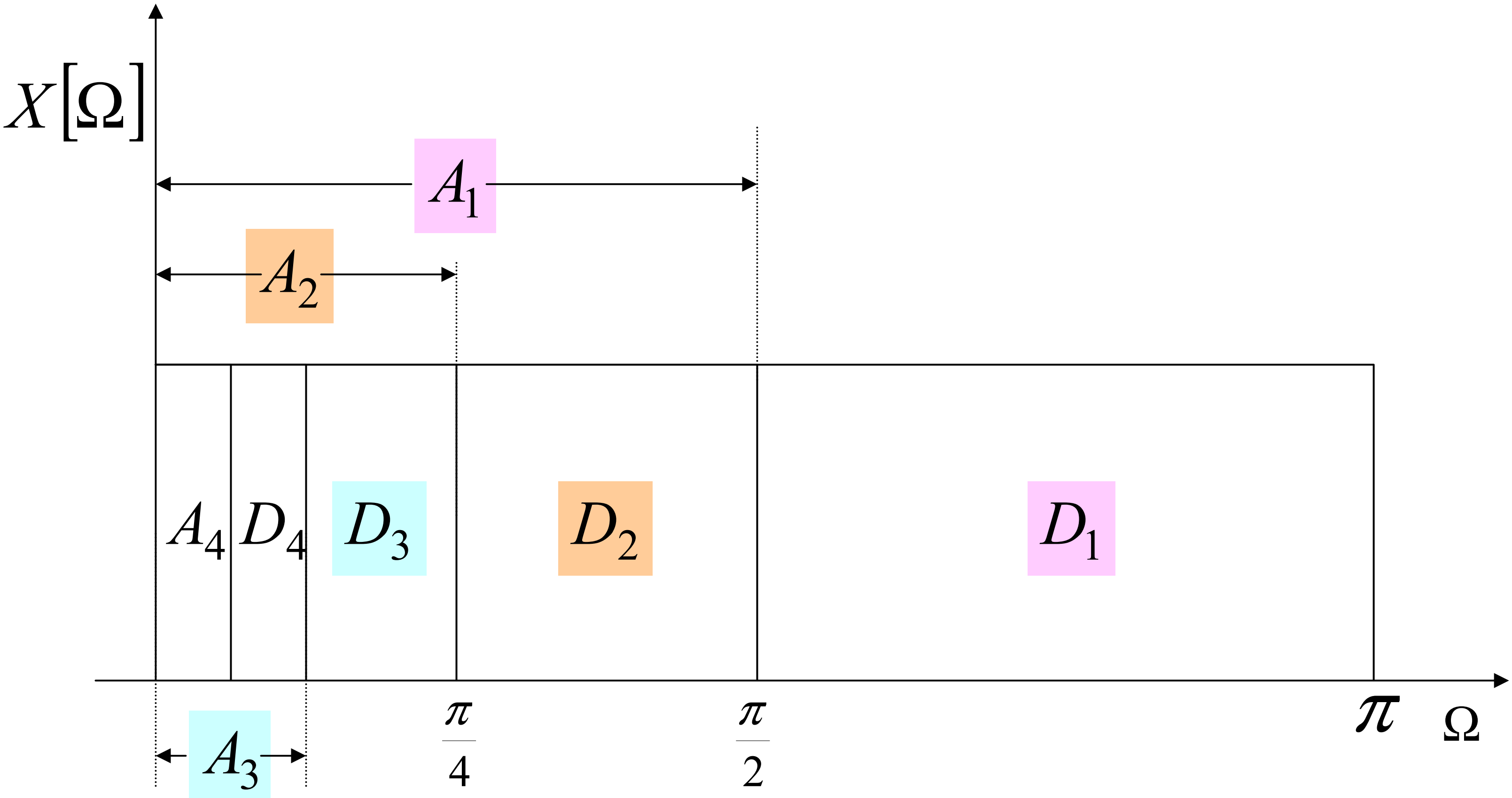
$$Gf = \sum_{k=-\infty}^{\infty} f[k]g[2n-k]$$

$$Hf = \sum_{k=-\infty}^{\infty} f[k]h[2n-k]$$



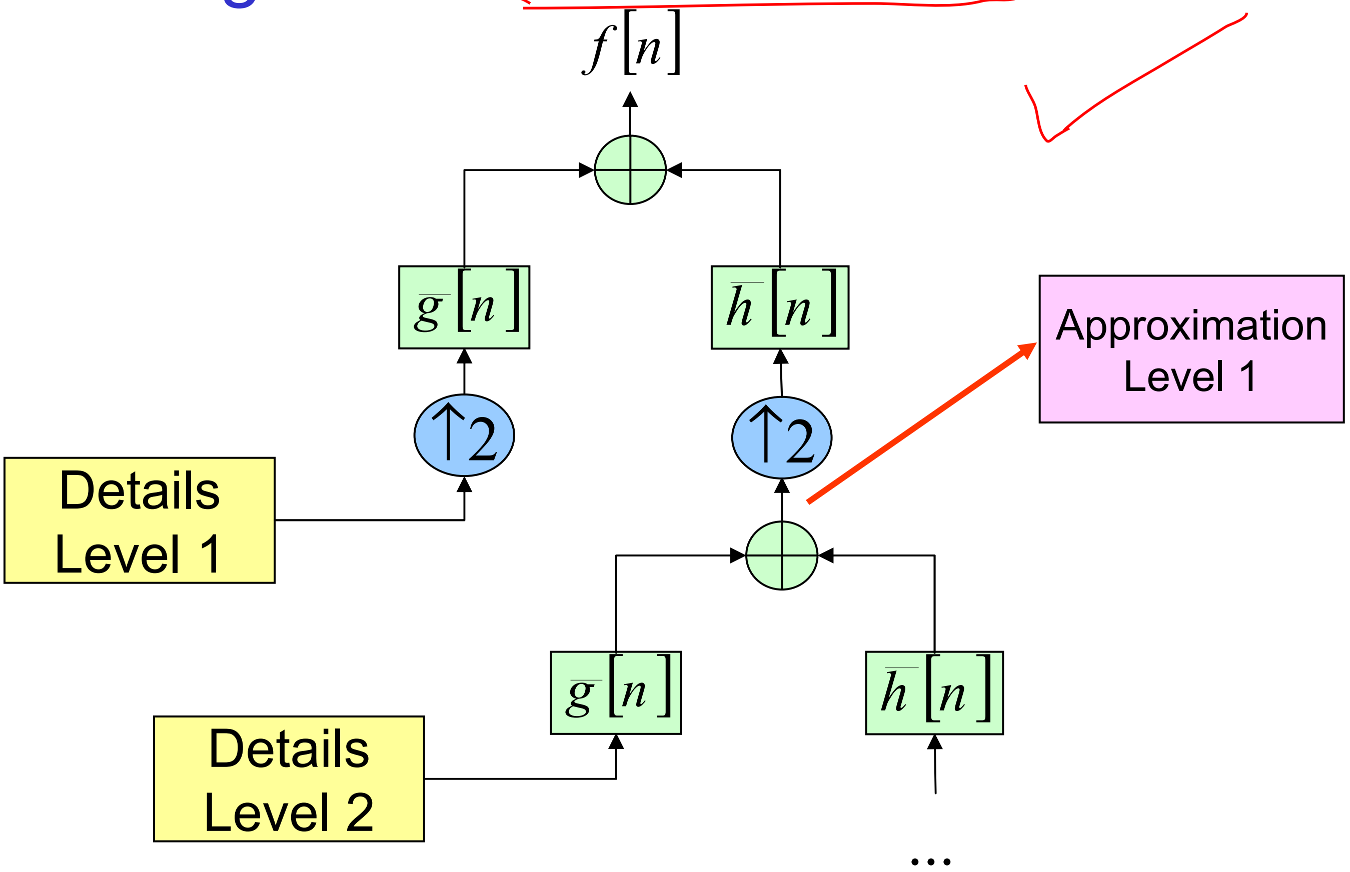
4.- Discrete Wavelet Transform.

- Pyramid Algorithm:



4.- Discrete Wavelet Transform.

- Pyramid Algorithm: Reconstruction.



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5.1.- Denoising.

- Finite Length Signal with Additive Noise:

$$y[n] = x[n] + \sigma\, r[n], 0 \leq n \leq N-1$$

Observations of x + noise

Signal

Standard Deviation

Noise : $r[n] \sim N(0,1)$

5.1.- Denoising.

In the Transformation Domain:

$$Y = X + \sigma R$$

where: $Y = Wy$ (W transform matrix).

$\hat{X}$ estimate of X from Y

Diagonal linear projection:

$$\Delta = diag\left(\delta_0, \delta_1, \ldots, \delta_{N-1}\right), \delta_i \in [0,1]$$

Estimate: $\hat{x} = W^{-1}\hat{X} = W^{-1}\Delta Y = W^{-1}\Delta W y$

5.1.- Denoising.

General Method:

1) Compute $Y = Wy$
2) Perform Thresholding in the Wavelet Domain.
3) Compute $\hat{x} = W^{-1}\hat{X}$

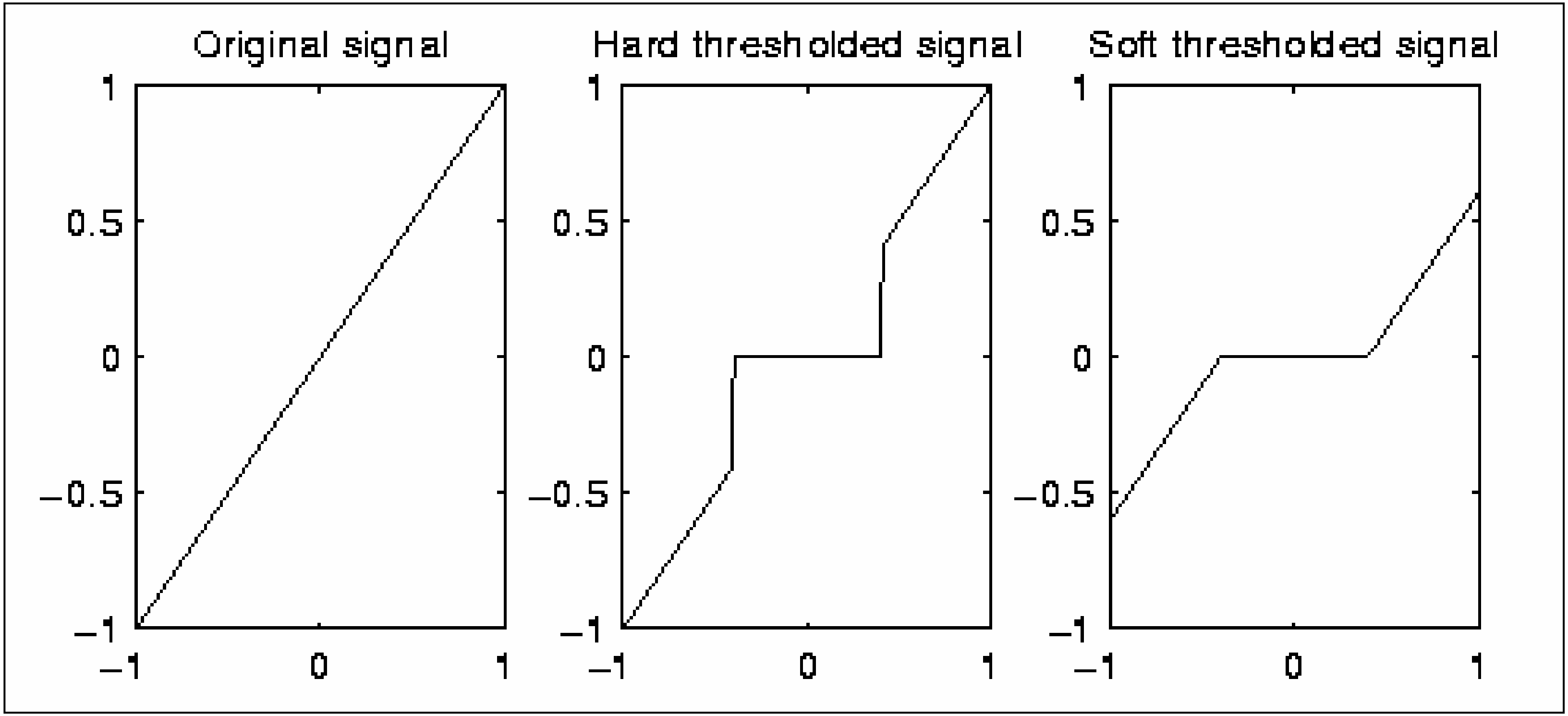
Questions:

Which Thresholding Method?

Which Threshold?

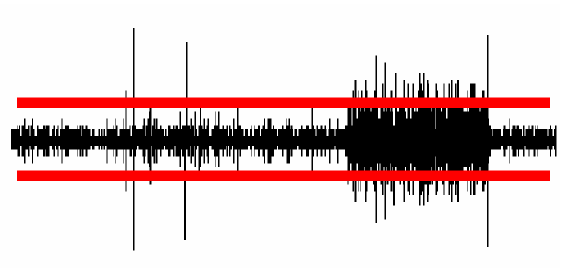
5.1.- Denoising.

Thresholding Method (Diagonal Linear Projection):



5.1.- Denoising.

Threshold:



Stein's Unbiased Risk Estimate (SURE).

Donoho: $\lambda = \sigma \sqrt{2 \log(n)}$ n:Sample size

σ:noise scale

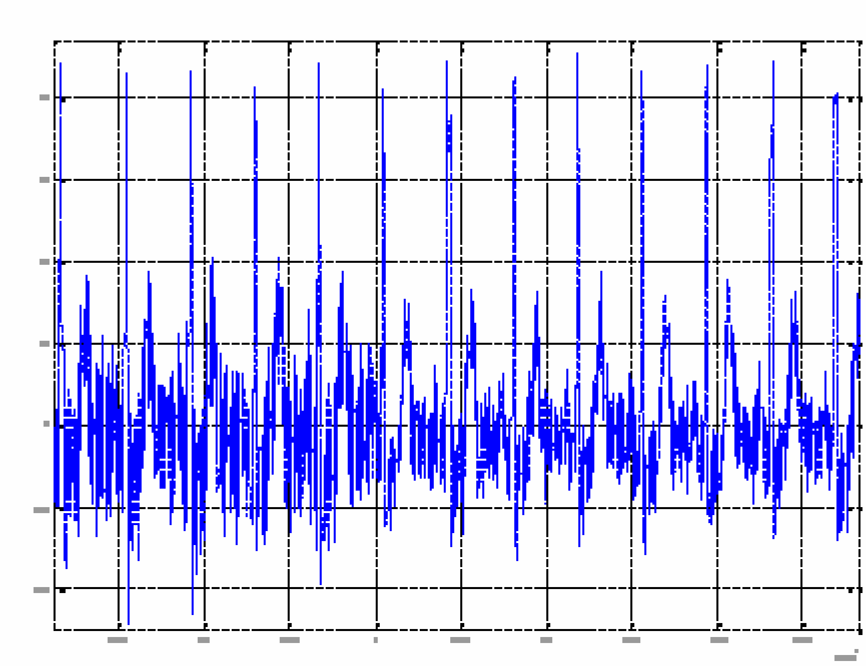
Minimax

Adaptive: based on singular or smooth regions (in practice, heuristic indep. of sample size)

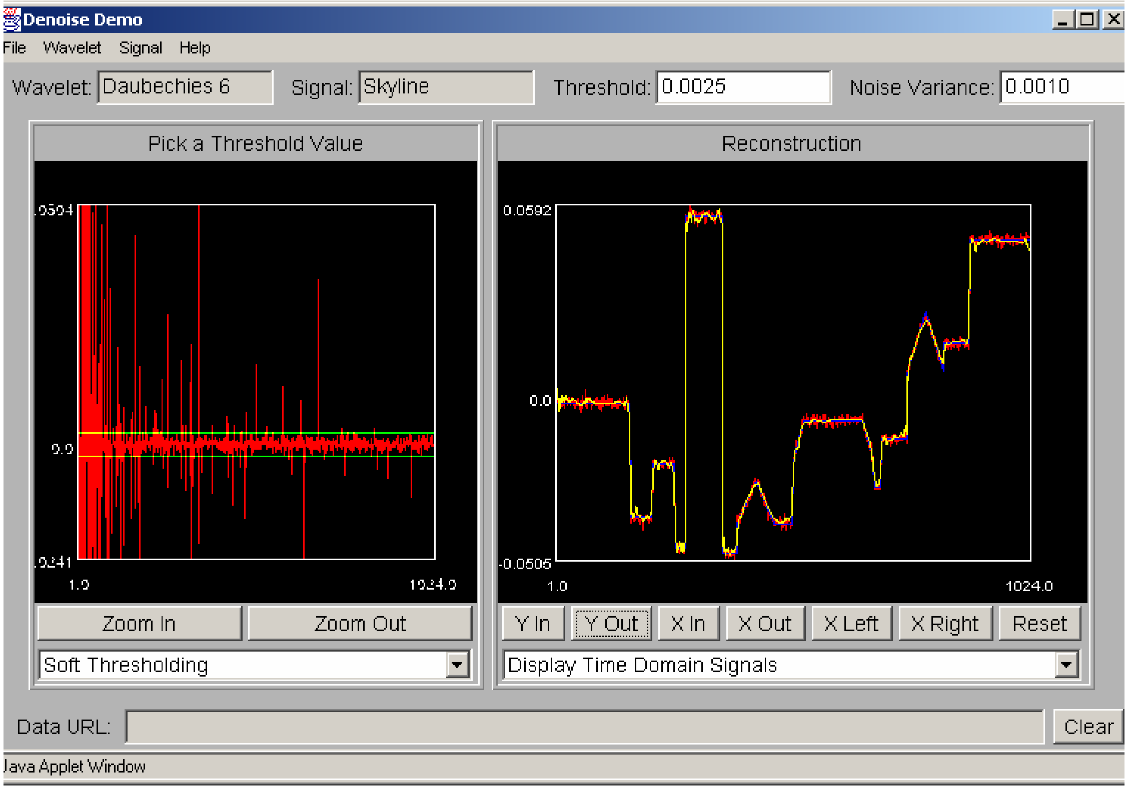
5.1.- Denoising.

Practical considerations:

- Signal normalization to avoid amplitude influence over the process.
- Noise can vary along the signal: windowed threshold.



5.1.- Denoising.



http://www-dsp.rice.edu/edu/wavelet/denoise.html

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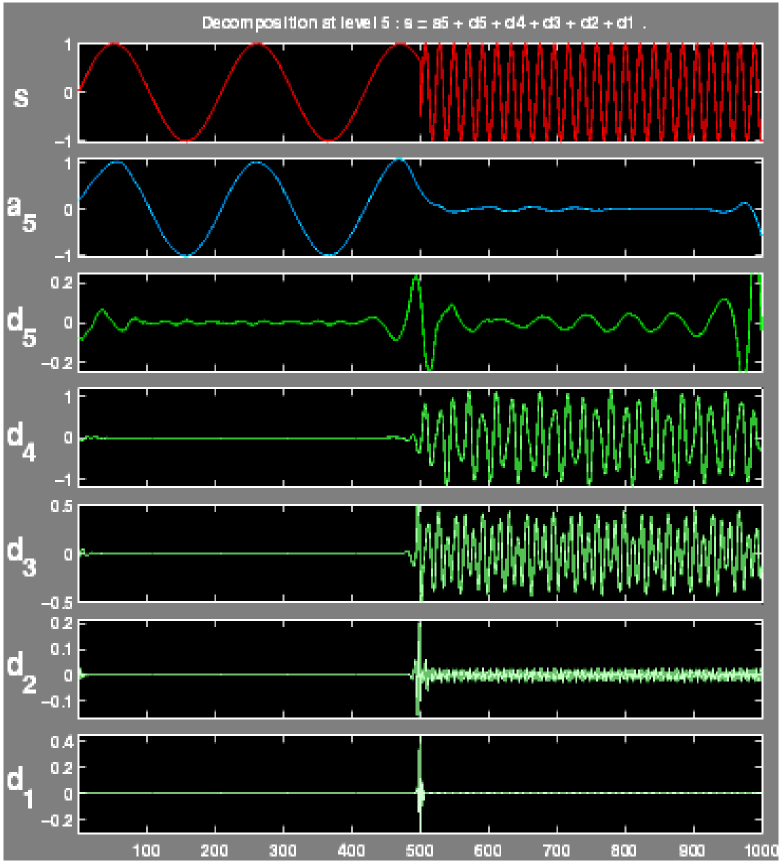
5.2.- Abrupt Change Detection.

- Low level details contain information of high frequency components.
- Small support wavelets are better suited to detect such changes.
- Noise makes it difficult.
- Application: QRS complex detection.



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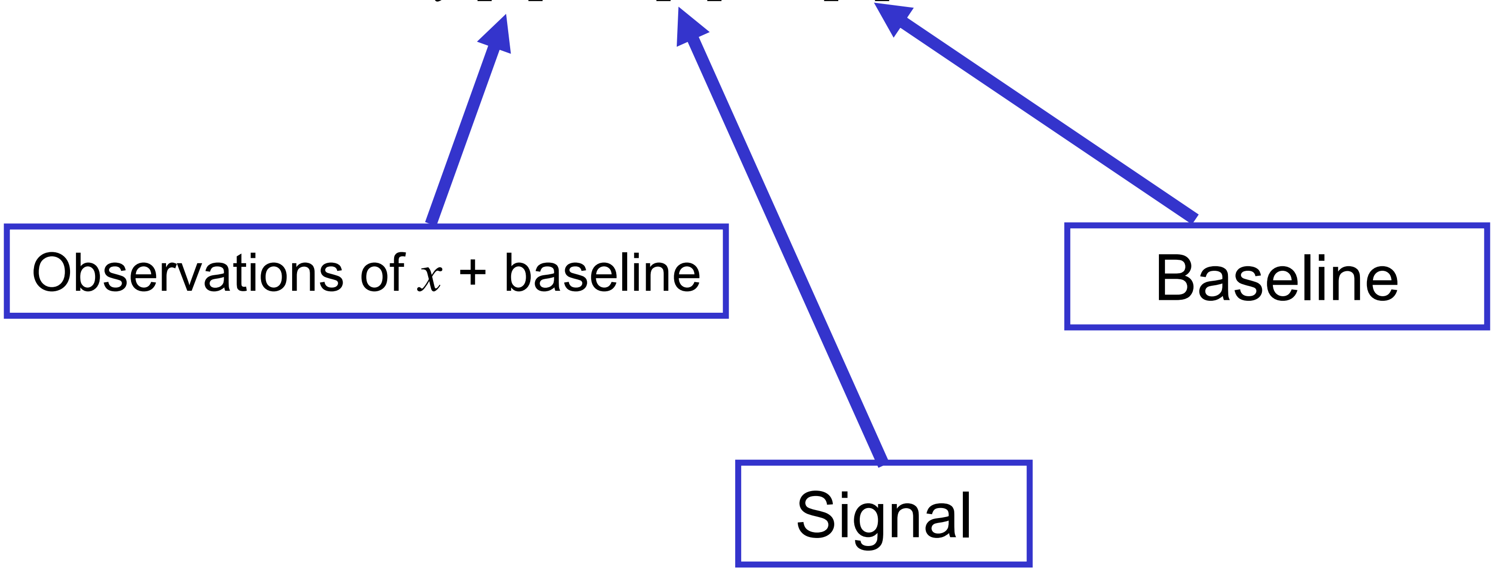
5.3.- Long-Term Evolution.

- Foundations:
  - Slow changes which take place in the signal, i.e., low frequency variations.
  - *In the time domain*: Wavelet transform has high resolution in time at low frequencies.
  - *In the frequency domain*: the higher the approximation level, the narrower the low pass filter is.
  - Problem: depending on the baseline(trend) frequency, the approximation level may change.
  - The baseline is considered as additive ‘noise’.

5.3.- Long-Term Evolution.

- Finite Length Signal with baseline variation:

$$y[n] = x[n] + b[n], 0 \le n \le N-1$$



5.3.- Long-Term Evolution.

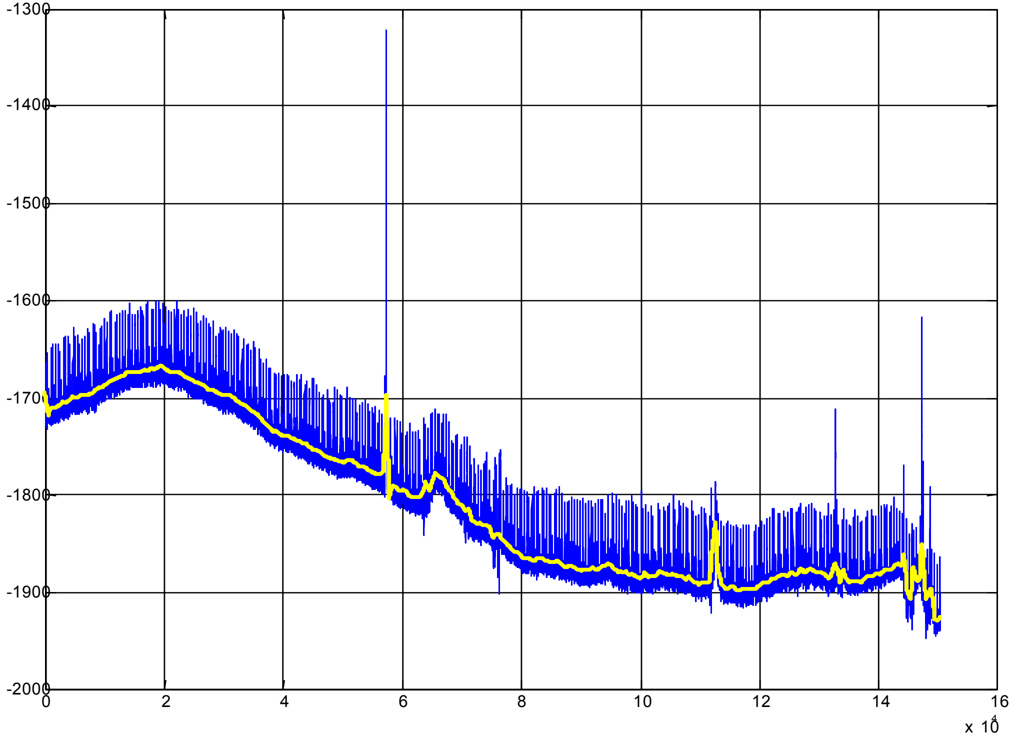
- With the Wavelet Approximation of certain level, we estimate the baseline.
- It can be used as extra information about the signal or to reduce the baseline oscillation.

$$WT\left(y[n]\right) = \hat{b}[n]$$

$$\hat{x}[n] = y[n] - \hat{b}[n]$$

5.3.- Long-Term Evolution.

- Application: Electrocardiogram baseline wandering reduction.



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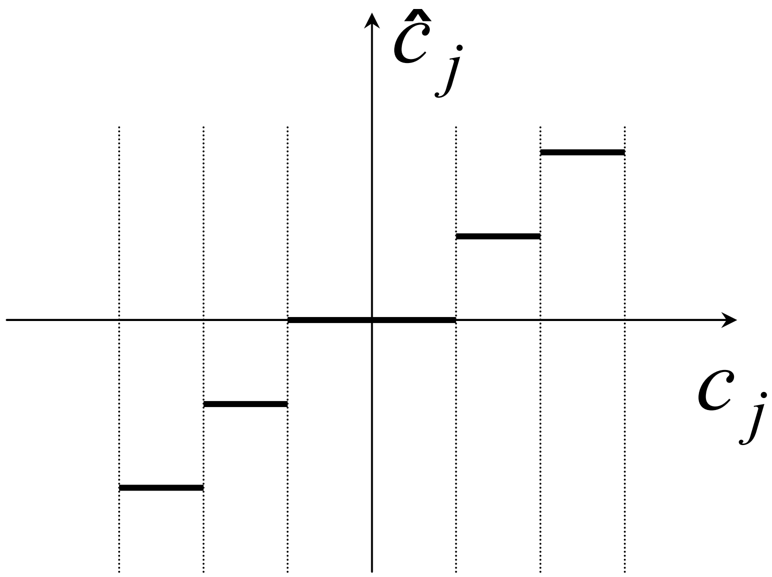
5.4.- Compression.

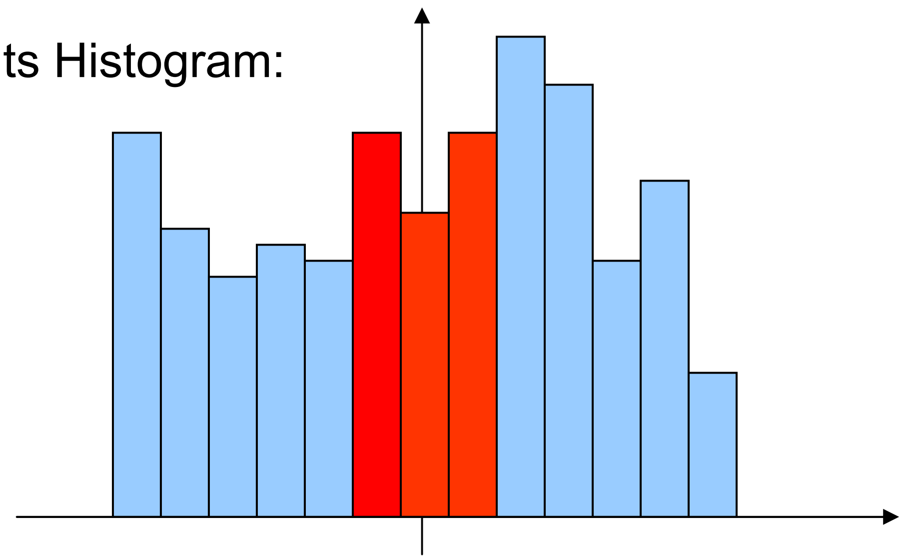
- Methods:
  - Quantization in the Wavelet domain. The set of possible coefficients values is reduced:



5.4.- Compression.

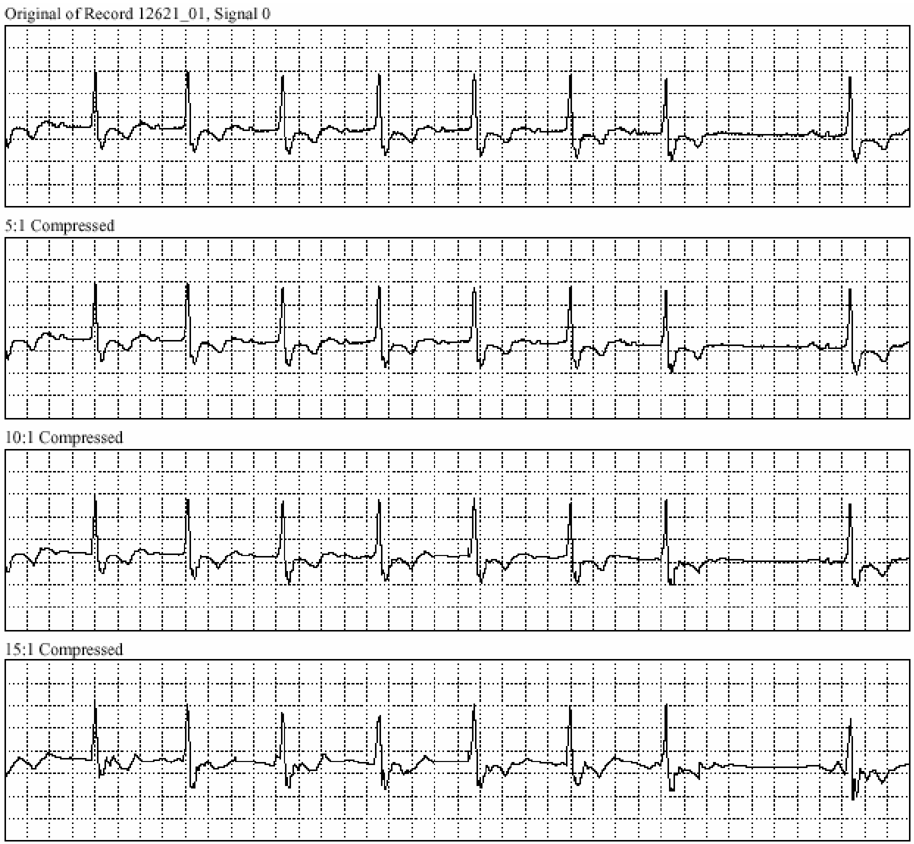
- Methods:
  - Discard those coefficients considered 'insignificant': Threshold.
  - Basis choice is very important.

Coefficients Histogram:



5.4.- Compression.

- Application: ECG compression.



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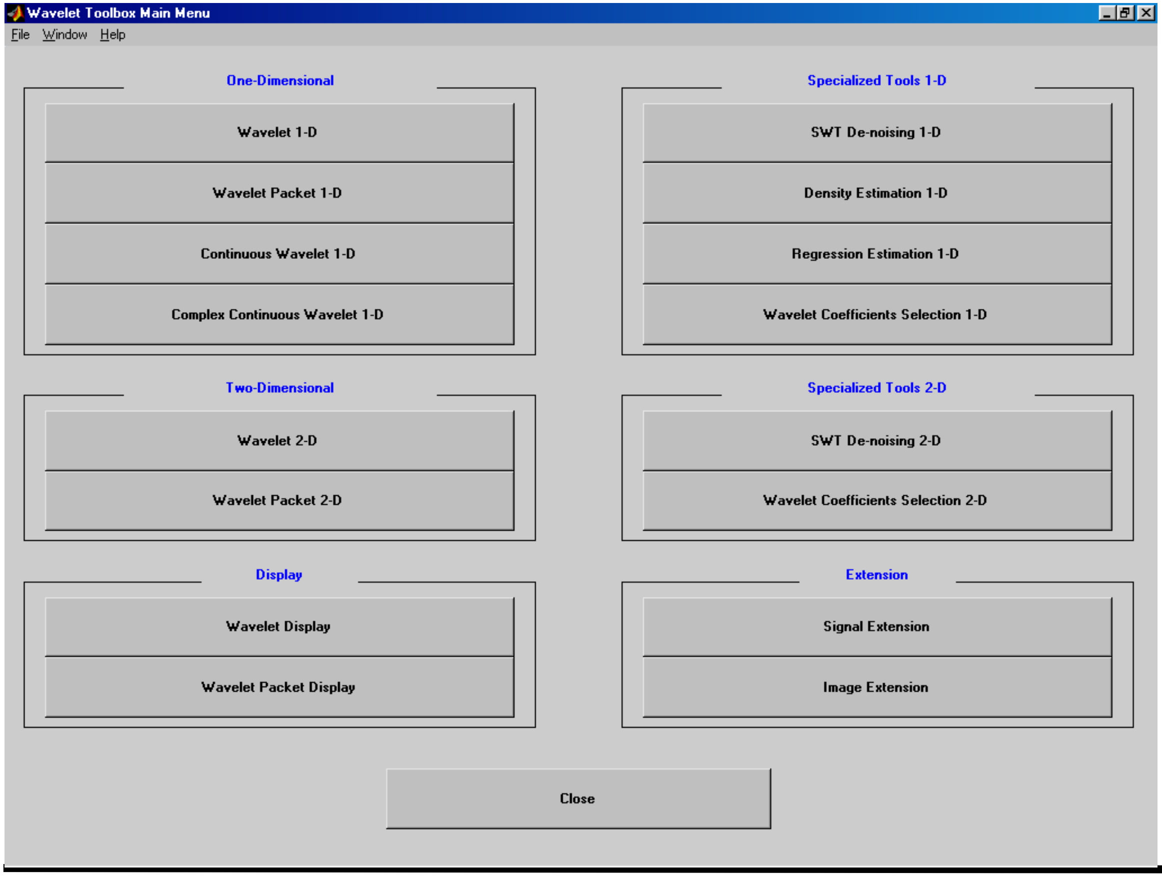
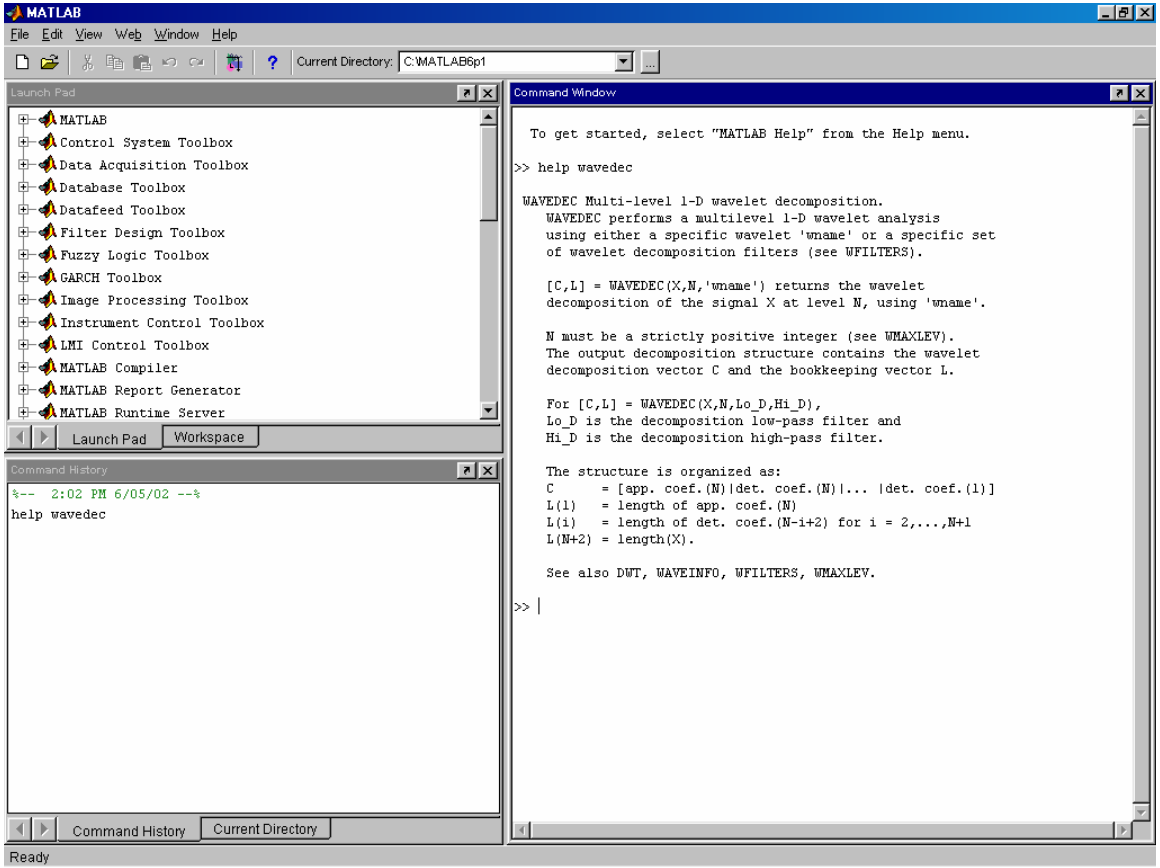
6.- The Matlab Wavelet Toolbox.

Matlab: GUI (type `wavemenu`).



6.- The Matlab Wavelet Toolbox.

Matlab: Command Line.



6.- The Matlab Wavelet Toolbox.

Functions:

DWT: Single-level 1-D wavelet decomposition.

```
[CA,CD] = DWT(X,'wname');
```

Approximation Coefficients

Detail Coefficients

Signal

Mother Wavelet

Example:

```
X=[1 1 1 0 0 0 1 1 1 0 0 0 1 1 1 1];

[CA,CD] = DWT(X,'haar');
```

```
CA = 1.4142     0.7071   0        1.4142   0.7071   0        1.4142
1.4142

CD = 0   0.7071   0        0        0.7071   0        0        0
```

Author: David Cuesta Frau

6.- The Matlab Wavelet Toolbox.

Functions:

IDWT: Single-level 1-D wavelet reconstruction.

```
X = IDWT(CA,CD,'wname');
```

WAVEINFO: Provides information about the wavelets in the toolbox.

```
WAVEINFO(‘wname');
```

DWTMODE: Sets extension mode to handle border distortion.

```
DWTMODE(‘status');
sym,zpd,sp0,sp1,ppd
```

6.- The Matlab Wavelet Toolbox.

Functions:

WAVEDEC: Multiple-level 1-D wavelet decomposition.

```
[C,L] = WAVEDEC(X,N,'wname')
```

APPCOEF: Extracts 1-D approximation coefficients.

```
A = APPCOEF(C,L,'wname',N)
```

DETCOEF: Extracts 1-D detail coefficients.

```
D = DETCOEF(C,L,N)
```

6.- The Matlab Wavelet Toolbox.

Functions:

WRCOEF: Reconstructs single branch from 1-D coefficients.

```
X = WRCOEF('type',C,L,'wname',N)
```

WDEN: Automatic 1-D denoising using wavelets.

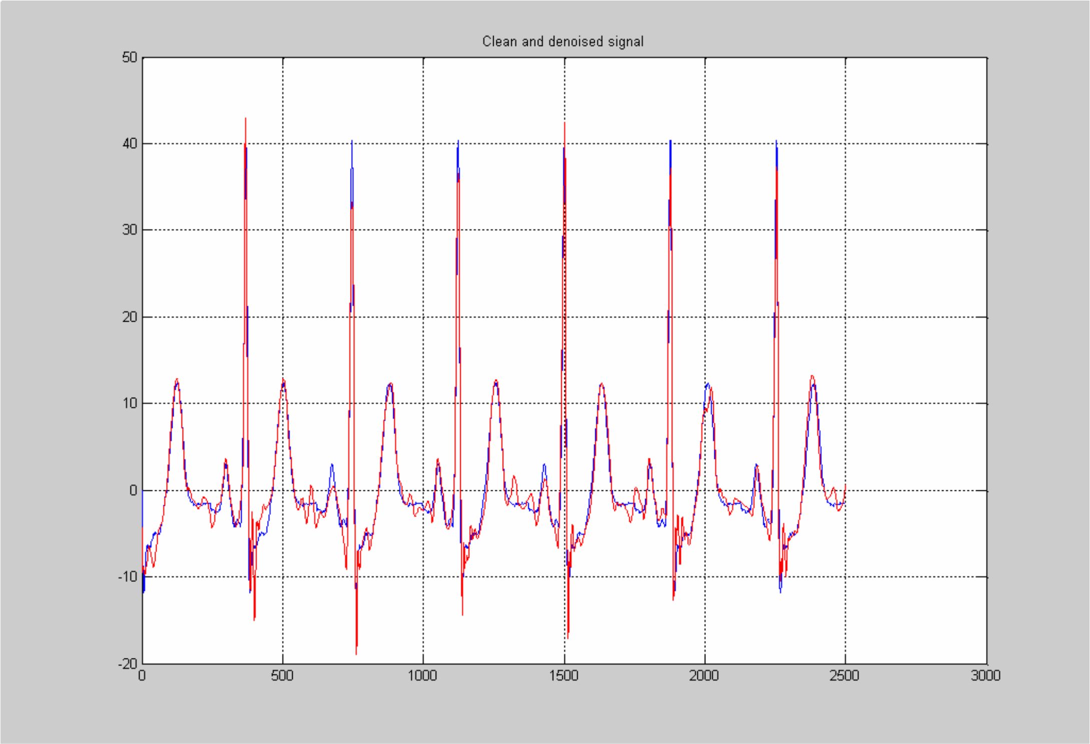
```
[XD,CXD,LXD] = WDEN(X,TPTR,SORH,SCAL,N,'wname')
```

- XD: Denoised signal
- X: Noisy signal
- TPTR: Threshold selection rule: `rigrsure` `heursure` `sqtwolog`
- SORH: Soft 's' or hard 'h' thresholding
- SCAL: Threshold rescaling `one` `sln` `mln`
- N: Decomposition Level
- 'wname': Mother Wavelet

6.- The Matlab Wavelet Toolbox.

Example:

Denoising an ECG signal using Wavelets (WaveletExample).



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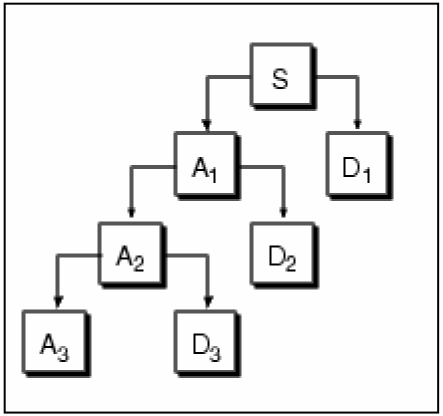
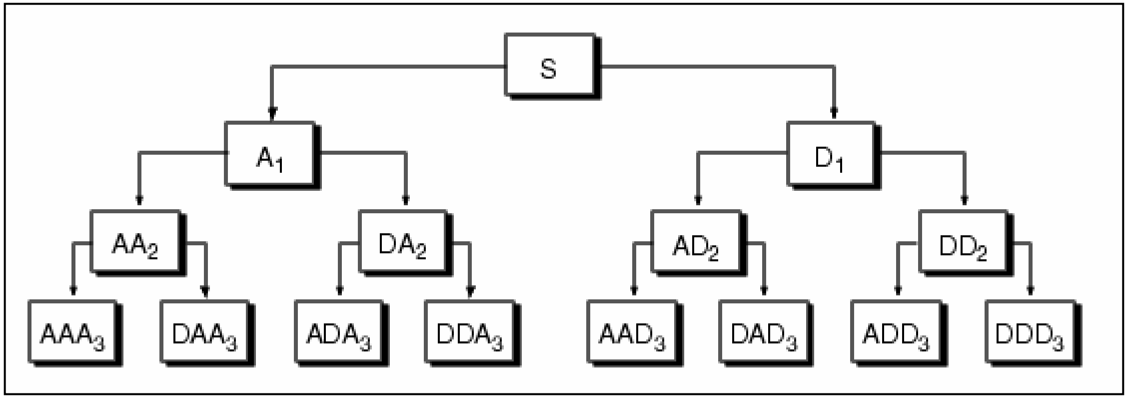
7.- Other Topics.

Wavelet Packets.



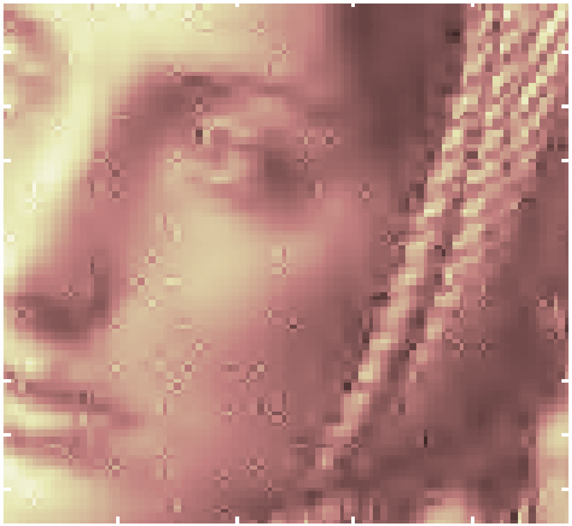


7.- Other Topics.

2D Wavelet Transform.

Application to bidimensional signals: images. Noise reduction, compression, etc.

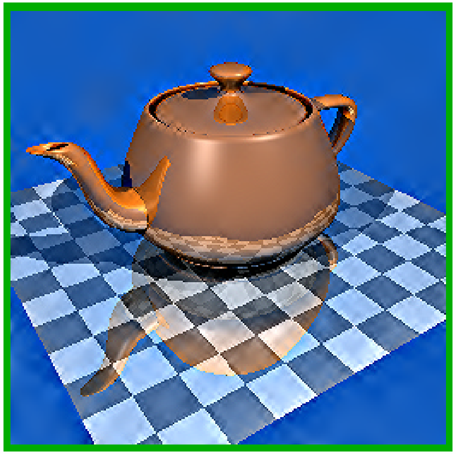
Image is considered a matrix: Wavelet transform is applied on rows and columns.



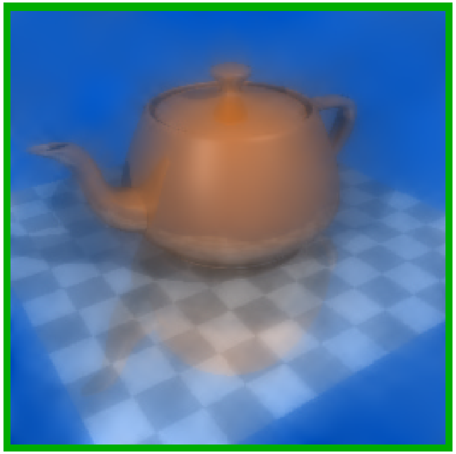
7.- Other Topics.

Computer Graphics.

- Image illumination computation.
- Figure animation.
- Surface modelling.
- Image compression.
- Image query.
- Etc.



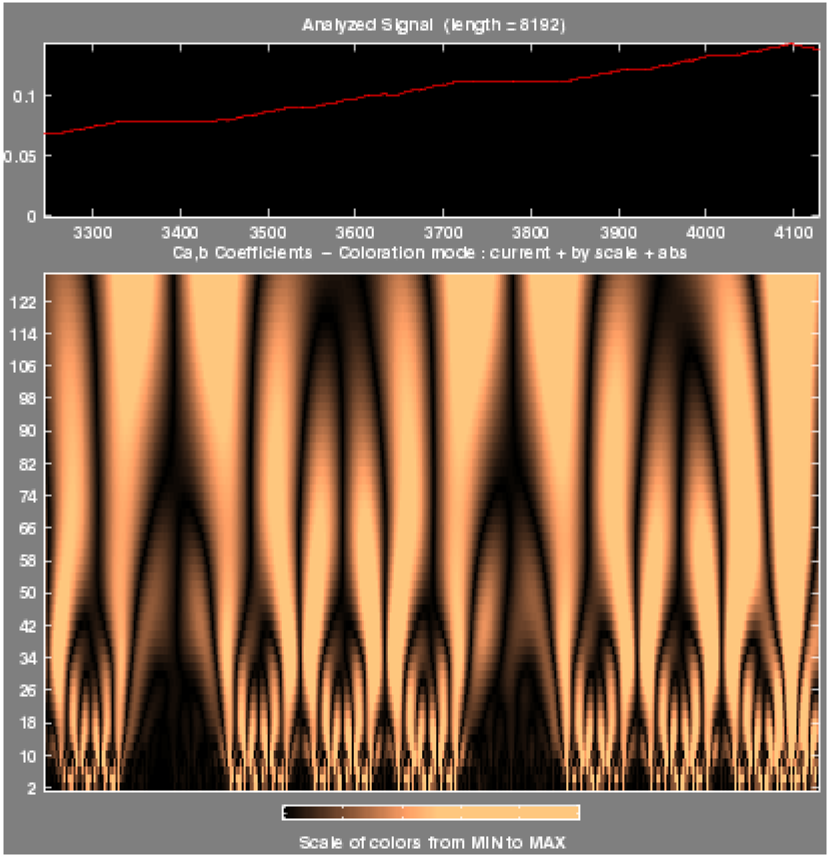
Scaling up wavelet coeffs.



Attenuating wavelet coeffs.

7.- Other Topics.

Self-Similarity Detection(Fractals).



7.- Other Topics.

Feature Extraction.

$$DWT\left(f[n]\right)=\begin{cases} c_k = \left\langle f[n],\varphi_k[n]\right\rangle = \sum_n f[n]\varphi_k[n] \\ d_{jk} = \left\langle f[n],\psi_{jk}[n]\right\rangle = \sum_n f[n]\psi_{jk}[n] \end{cases}$$

Approximation of the input signal

$$f[n]=\{f[0],f[1],\ldots,f[N-1]\}\Rightarrow c_k\left(|c_k| << N\right)$$

7.- Other Topics.

Applications in electrocardiology:

- ECG Compression.
- ECG Pattern Recognition: detection and classification of ECG waves, discriminating normal and abnormal cardiac patterns.
- ST segment analysis. Change detection.
- Heart rate variability: ECG is assumed to be nonstationary.
- High resolution electrocardiography: detection of late potentials.

Index

8.- Summary.

- Fourier transform is not suitable to obtain time and frequency information: Wavelet Transform.
- Multiresolution analysis. It is possible to change time and scale.
- Discrete Wavelet Transform. Pyramidal Algorithm: Efficient algorithm to calculate the Wavelet Transform in a computer.
- Applications in signal processing: noise reduction, long-term evolution, compression, abrupt changes.
- Very important to the analysis of biological signals since most of the statistical characteristics of such signals are nonstationary-
- Matlab Wavelet Toolbox: definitions and functions to work with Wavelets.
- Many other topics: Wavelet packets, Computer graphics, etc.

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9.- Bibliography.

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Signal Processing

Introduction to Wavelets

- Laboratory Exercises

- Assignments